ADAPTIVE OPTIMAL CONTrollable AND OBSERVABLE PAIRING BY THE ASSIGNMENT METHOD BASED ON THE BALANCE REALIZATION MEASURE

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ABSTRACT

Pairing is an important task in controlling of a MIMO plant by some SISO sub-control systems. Using balance realization is one of the methods to find the best pairing. In this article, an algorithm is proposed to find an optimal controllable and observable input-output pairing. The algorithm can utilize when the plant parameters change and propose the new pairing to adapt the control structure.

INTRODUCTION

Large scale MIMO plant control by a MIMO controller is a troublesome task which can not necessarily propose a good control performance due to the complexity of the controller design method. Therefore, it is better to consider the MIMO plant as some SISO ones by pairing the inputs and the outputs and design a SISO control system for each of this subsystems. In this case the effect of the other inputs into the present output is considered as a disturbance. To have a better performance, it is necessary to minimize this disturbance, as another source of disturbance. Therefore having an optimal pairing is crucial for this problem. There are some pairing methods proposed in literature like RGA, Bristol (5), and its deviations like RIA, Zhong (6). Another class of pairing is based on the cross-gramian matrix like balance realization, Khaki-Sedigh and Shahmansourian (2), and participation matrix, Conley and Salgado (3), methods. In all of them a kind of matrix is obtained based on some criteria of the pairing quality and then the pairs are selected to optimize some measure in that matrix, subject to some conditions.

The Cross-Gramian matrix is a measure to quantify the controllability and observability of a plant. This matrix is used in (2) to introduce an algorithm for pairing of a MIMO plant. The algorithm is an applicable measure for a good pairing; however, it can not guaranty offering an optimal controllable and observable pairing without need of some human decision (2). In this paper we introduce an algorithm to find an adaptive optimal controllable and observable pairing, which can utilize to find the pairing in a variable plant adaptively. The algorithm is based on the optimization selection method called assignment method known in the theory of the operations research, Cooper and Steinberg (4).

In the next section the balance realization method is reviewed. Then in section 3 the new algorithm is introduced to find an optimal realization, which can be used in an adaptive pairing. In the subsequent section the algorithm is modified to find an optimal controllable and observable pairing. The results are concluded in the last section.

BALANCED REALIZATION BASED PAIRING

For a system with a state space representation \((A,B,C,0)\), there are controllability and observability gramian matrix \(P\) and \(Q\), which are defined by:

\[
AP + PA^T + BB^T = 0 \tag{1}
\]

\[
A^TQ + QA + C^TC = 0 \tag{2}
\]

The matrix \(W_{co} = (PQ)^{1/2}\) is known as cross gramian matrix which is independent of the system representation. This matrix is a quantitative measure of controllability and observability of the system. As the determinant of \(W_{co}\) increases, the controllability and observability of the system will increase. For a multivariable \(n\times n\) system \(S:(A,B,C,0)\) with subsystems \(S_i:(A_i,b_i,c_i,0)\) let’s call \(P_i\) and \(Q_i\) the controllability and observability gramian matrices of the subsystem \(S_i\). Then the matrix \((W_{co}) = (P_iQ_i)^{1/2}\) is a measure of controllability and observability of the subsystem.

Khaki-Sedigh and Shahmansourian (2) used the determinant of the matrix \((W_{co})\) and suggested the following algorithm for pairing of a MIMO system to some SISO subsystems:

i) Find the matrix \(\Phi = \{\phi_{ij}\}\) where \(\phi_{ij} = \text{det}(W_{co})\) (3)

ii) Choose the largest element of \(\Phi\), say \(\varphi_{i*,j}\). Pair input \(i^*\) to output \(j^*\).

iii) Ignore the row \(i^*\) and the column \(j^*\) and continue from the step ii until the pairing is completed. This algorithm selects some better pairing for the early pairs but the later pairs could be poor. Conley and Salgado (3) propose pairing \(F^*\) such that

\[
\psi_{P_i} = \max(\psi_{P_j}, P_j \text{ is a complete pairing set}, \tag{4}
\]

where

\[
\psi_{P_i} = \sum_{\rho_j \neq P_i} \phi_{ij}, \tag{5}
\]

and \(\rho_j\) stands for the pairing \((u_i,v_j)\).

By this criterion the overall pairing is better, although some good pairs might not be selected. However, neither algorithm can reject the uncontrollable and/or unobservable pairs automatically (2), (3).
HUNGARIAN ALGORITHM IN ASSIGNMENT

Hungarian Algorithm is a systematic method to solve assignment problems. The algorithm is well explained in Cooper and Steinberg (4) and we just review its main idea.

In assignment problem one is to assign \( n \) objects \( x_i \in X \) to \( n \) objects \( y_j \in Y \) with a minimum cost. The cost of assigning \( x_i \) to \( y_j \) is \( c_{ij} \). To solve the problem the matrix \( C=[c_{ij}] \) is defined. Therefore, the problem is selection of \( n \) elements of \( C \) such that 1) there is one and only one selection in each row/column, 2) sum of the selected elements is minimal. It is proved that adding a constant value to a row or column does not affect the optimal assignment. Therefore the algorithm tries to add some negative values to the rows/columns such that some zeroes appeared in the matrix but none of them become negative. In this case if an assignment can be obtained among the zero elements it is the optimal assignment. The Hungarian algorithm makes zeroes and then makes an assignment among them.

OPTIMALPAIRING

Suppose finding the \( n \) pairs of \( P^* \) in the system \( S \) that maximize the function (5). The algorithm to obtain the optimal pairing can be proposed as follows.

Algorithm 1. Step1) Define \( \alpha \) as
\[
\alpha = \max_{i,j} \varphi_{ij}
\]
(6)

Step 2) Define the new matrix \( \Theta \) as
\[
\Theta = \{ \vartheta_{ij} \}, \vartheta_{ij} = \alpha - \varphi_{ij}
\]
(7)
It is obvious that all the elements of \( \Theta \) are nonnegative. Maximizing the function \( \psi \) is equivalent to minimizing the function
\[
\chi_{\varphi} = \sum_{i=1}^{n} p_i \vartheta_{ij},
\]
(8)

Step 3) Use Hungarian Algorithm (4) to minimize equation (8).

The problem of step (2) is a classical assignment problem known in the theory of operations research. The Hungarian Algorithm solves this problem and the optimal pairs can be found systematically.

Example 1. Consider example 1 of (2) with the plant
\[
G(s) = \frac{1-s}{s^2 + s + 1}
\]
(9)

Using balance realization matrices, the matrix \( \Phi \) is:
\[
\Phi = \begin{bmatrix}
0.0081 & 11.8915 & 0.0081 \\
3.4964 & 0.0081 & 0.0081 \\
3678.7 & 3687.7 & 0.0081
\end{bmatrix}
\]
(10)
The method of (2) proposes two possible pairings
\[
P_1 = (p_{12}, p_{23}, p_{31}) \text{ with } \psi_1 = 3690.6.
\]

Using the above optimal pairing method, the matrix \( \Theta \) is:
\[
\Theta = \begin{bmatrix}
3678.7 & 3666.8 & 3678.7 \\
3676.2 & 3678.7 & 3678.7 \\
0 & 0 & 3678.7
\end{bmatrix}
\]
(11)

Applying the Hungarian algorithm to the above matrix obtains the new matrix
\[
\Theta_a = \begin{bmatrix}
11.9 & 0 & 0 \\
9.4 & 11.9 & 0 \\
0 & 0 & 3666.8
\end{bmatrix}
\]
(12)
which offers the optimal pairing
\[
P^* = (p_{12}, p_{23}, p_{31}) \text{ with } \psi = 3690.6.
\]

Example 2. Consider the plant
\[
G(s) = \frac{1}{s^2 + s + 1}
\]
(13)

Based on the balance realization matrices, the matrix \( \Phi \) is:
\[
\Phi = \begin{bmatrix}
0.1501 & 0.256 & 0.4101 & 0.9151 & 0.81 & 0.625 \\
0.256 & 0.4101 & 0.256 & 1.296 & 0.9151 & 0.4101 \\
0.4101 & 0.1501 & 1.296 & 0.256 & 0.256 & 0.256 \\
0.625 & 0.9151 & 1.296 & 0.1501 & 0.9151 & 0.256 \\
0.1501 & 0.081 & 0.9151 & 0.4101 & 0.256 & 0.4101 \\
0.4101 & 0.1501 & 0.625 & 0.625 & 0.256 & 1.296
\end{bmatrix}
\]
(14)

Using the method of (2), there are three possible pairings
\[
P_1 = (p_{12}, p_{23}, p_{31}, p_{35}, p_{56}) \text{ with } \psi_1 = 5.2902 \times 10^3,
\]
\[
P_2 = (p_{12}, p_{23}, p_{35}, p_{56}) \text{ with } \psi_2 = 5.2902 \times 10^3\text{and}
\]
\[
P_3 = (p_{12}, p_{23}, p_{35}, p_{56}) \text{ with } \psi_3 = 4.8101 \times 10^3.
\]
But, using the above optimal pairing method the matrix \( \Theta \) is:
\[
\Theta = \begin{bmatrix}
1.1459 & 1.04 & 0.8859 & 0.3809 & 1.215 & 0.671 \\
0.8859 & 0.1459 & 0 & 0.3809 & 0.3809 & 0.8859 \\
0.8859 & 1.1459 & 0 & 0.3809 & 0.3809 & 0.3809 \\
0.8859 & 0.1459 & 0.8859 & 0.3809 & 0.3809 & 0.1459 \\
0.8859 & 1.1459 & 0.3809 & 0.3809 & 0.3809 & 0.3809 \\
0.8859 & 0.1459 & 0.8859 & 0.3809 & 0.3809 & 0.8859
\end{bmatrix}
\]
(15)

Applying the Hungarian algorithm to the above matrix obtains the new matrix.
For any pairing

\[ P^* = \{p_{14}, p_{25}, p_{31}, p_{45}, p_{45}, p_{56}\} \]  

with \[ \psi = 5.4824 \times 10^3 \]

which has a better overall pairing according to the criterion (4).

The Hungarian algorithm for assignment problem is a systematic solution. Therefore, the above method finds the optimal selection automatically and it can be used in adaptive pairing. To do this, as the parameters of the plant change, the \( \Phi \) matrix is computed and the optimal pairs can obtain online.

**CONTROLLABLE AND OBSERVABLE PAIRING**

Controllability and observability of the pairs is crucial in dividing the MIMO plants into some SISO pairs. None of the above techniques can guarantee obtaining controllable and observable pairs. Using the balance realisation matrix, a pair \( S_j \) is uncontrollable and/or unobservable if \( \phi_j \) is zero. In the method (2), \( \phi_j \) of the last assigned pairs might be zero. To avoid it, the designer shall change some of the pairs by trial and error to some new pairs so that none of the \( \phi_j \) are zero and the pairing performance is still acceptable.

Using the above optimal pairing it is possible to avoid uncontrollable and/or unobservable pairs automatically. For this purpose, let’s consider the following lemma.

**Lemma 1.** If a value

\[ \beta \geq \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} \]  

(17)

adds to an element \( \theta_k \) of the matrix \( \Theta \), then the pair \( p_k \) is not selected in the optimal pairing by the Algorithm 1.

**Proof:**

For any pairing \( P \)

\[ \psi = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} \leq \beta \]  

(18)

On the other hand

\[ \psi = n\alpha - \chi_P \]  

(19)

Where \( \alpha \) defines in (6). Therefore

\[ n\alpha - \chi_P \leq \beta \quad \Rightarrow \quad \chi_P \geq n\alpha - \beta \]  

(20)

Suppose that \( p_\alpha \) is an undesired pair, for instance uncontrollable and/or unobservable pair. If \( p_\alpha \) is not selected, then adding \( \beta \) to \( \theta_k \) changes \( \chi_P \) to

\[ \chi_P^m = \chi_P + \beta, \quad \chi_P^m \geq n\alpha \]  

(21)

On the other hand, for any pairing \( P \) which does not include \( p_\alpha \):

\[ \chi_P \leq n\theta_{\text{max}} \]  

(22)

where

\[ \theta_{\text{max}} = \theta_{\text{min}} \]

Since \( \phi_0 \geq 0 \), then \( \theta_{\text{max}} \leq \alpha \), that is \( \chi_P \leq n\alpha \)

(23)

Therefore

\[ \chi_P^m \geq \chi_P \]  

(24)

It means \( P_\alpha \) is not optimal, that is the pair \( p_\alpha \) does not include to the optimal pairing set \( P \). Using Lemma 1, the Algorithm 1 can be modified as the following to avoid selection of any uncontrollable and/or unobservable pairs.

**Algorithm 2.** Step 1 & 2) Same As step 1 & 2 of Algorithm 1.

**Step 3)** Choose \( \beta \) such that satisfies equation (12).

**Step 4)** Add \( \beta \) to any \( \theta_k \) where \( \phi_k = 0 \).

**Step 5)** Apply the Hungarian algorithm as step 3 of Algorithm 1.

The above method can be used to avoid selection of any other pairs which is not to be selected in the pairing, due to any other reasons like geometrical distances between the variables.

**Example 3.** Suppose the plant of example 2 of (2):

\[ G(s) = \frac{2}{(s+1)(s+2)} \begin{bmatrix} s+1 & s & s+2 \\ 0 & -1 & s+3 \\ 0 & 0 & s+4 \end{bmatrix} \]  

(26)

The matrix \( \Phi \) is

\[ \Phi = \begin{bmatrix} 0.125 & 0 & 0 \\ 7.716 \times 10^{-4} & 1.929 \times 10^{-4} & 0 \\ 0.25 & 7.716 \times 10^{-4} & 0.0069 \end{bmatrix} \]  

(27)

and therefore the matrix \( \Theta \) is

\[ \Theta = \begin{bmatrix} 0.125 & 0.25 & 0.25 \\ 0.24923 & 0.24981 & 0.25 \\ 0 & 0.24923 & 0.2431 \end{bmatrix} \]  

(28)

To avoid selection of \( p_{12}, p_{13}, p_{23} \), \( \beta \) could be

\[ \beta \geq \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} = 0.38364 \]  

(29)

This \( \beta \) shall add to the mentioned pairs in \( \Theta \). Therefore, the modified \( \Theta \) matrix, \( \Theta_m \), will be:

\[ \Theta_m = \begin{bmatrix} 0.125 & 0.63364 & 0.63364 \\ 0.24923 & 0.24981 & 0.63364 \\ 0 & 0.24923 & 0.2431 \end{bmatrix} \]  

(30)

Using the Hungarian method:

\[ \Theta_b = \begin{bmatrix} 0 & 0.25941 & 0.26554 \\ 0.24865 & 0 & 0.38996 \\ 0 & 0 & 0 \end{bmatrix} \]  

(31)

which offers the optimal controllable/observable pairing

\[ P^* = \{p_{11}, p_{22}, p_{33}\} \]
The matrix \( \Phi \) is
\[
\begin{bmatrix}
0.1501 & 0.256 & 0.4101 & 0.9151 & 0 & 0.625 \\
0.256 & 0.4101 & 0.256 & 1.296 & 0.9151 & 0.4101 \\
0.4101 & 0.1501 & 0 & 0.256 & 0.256 & 0.256 \\
0.625 & 0.9151 & 1.296 & 0.1501 & 0.9151 & 0.256 \\
0.1501 & 0.081 & 0.9151 & 0.4101 & 0 & 0.4101 \\
0.4101 & 0.1501 & 0.625 & 0.625 & 0.256 & 1.296
\end{bmatrix}
\]
which shows that the pairs \( p_{15}, p_{33} \) & \( p_{55} \) are uncontrollable and/or unobservable, as obvious from the plant transfer function (320). Using the method of (2) the pairs shall be \( \Phi = (p_{12}, p_{24}, p_{31}, p_{43}, p_{55}, p_{66}) \) which is uncontrollable/unobservable. To avoid this situation a way is to change the 3\(^{rd}\) and 5\(^{th}\) pairs and consider the pairing \( \Phi = (p_{12}, p_{24}, p_{35}, p_{43}, p_{51}, p_{66}) \) with \( \psi = 4.5501 \times 10^3 \).

Using the Algorithm 2, the matrix \( \Theta \) is
\[
\begin{bmatrix}
1.1459 & 1.04 & 0.8859 & 0.3809 & 1.296 & 0.671 \\
1.04 & 0.8859 & 1.04 & 0 & 0.3809 & 0.8859 \\
0.8859 & 1.1459 & 1.296 & 1.04 & 1.04 & 1.04 \\
0.671 & 0.3809 & 0 & 1.1459 & 0.3809 & 1.04 \\
1.1459 & 1.215 & 0.3809 & 0.8859 & 1.296 & 0.8859 \\
0.8859 & 1.1459 & 0.671 & 0.671 & 1.04 & 0
\end{bmatrix}
\]

To avoid selection of the above mentioned pairs \( \beta \) could be
\[
\beta \geq \sum_{i=1}^{6} \sum_{j=1}^{6} \phi_{ij} = 16.713 \times 10^3
\]

Therefore the modified \( \Theta \) matrix, \( \Theta_m \), will be:
\[
\begin{bmatrix}
1.1459 & 1.04 & 0.8859 & 0.3809 & 18.009 & 0.671 \\
1.04 & 0.8859 & 1.04 & 0 & 0.3809 & 0.8859 \\
0.8859 & 1.1459 & 18.009 & 1.04 & 1.04 & 1.04 \\
0.671 & 0.3809 & 0 & 1.1459 & 0.3809 & 1.04 \\
1.1459 & 1.215 & 0.3809 & 0.8859 & 18.009 & 0.8859 \\
0.8859 & 1.1459 & 0.671 & 0.671 & 1.04 & 0
\end{bmatrix}
\]

Applying the Hungarian algorithm obtains the new matrix \( \Theta_h = 10^3 \times \)
\[
\begin{bmatrix}
0.094 & 0.2781 & 0.505 & 0 & 17.2471 & 0.2901 \\
0.369 & 0.505 & 1.04 & 0 & 0 & 0.8859 \\
0 & 0.5501 & 17.7941 & 0.8251 & 0.4441 & 0.8251 \\
0 & 0 & 0 & 1.1459 & 0 & 1.04 \\
0.094 & 0.4531 & 0 & 0.505 & 17.2471 & 0.505 \\
0.2149 & 0.765 & 0.671 & 0.671 & 0.6591 & 0
\end{bmatrix}
\]

and offers the optimal pairing \( \Phi = (p_{12}, p_{24}, p_{31}, p_{43}, p_{55}, p_{66}) \) with \( \psi = 5.3665 \times 10^3 \).

The condition (12) for \( \beta \) is a sufficient condition and quit conservative. For example even applying the Hungarian algorithm to the matrix \( \Theta \), i.e. for \( \beta = 0 \), the same pairing is obtained, but the condition (12) guaranty avoidance of selecting any undesired pairs.

**CONCLUSION**

The problem of finding an optimal pairing for a MIMO system based on the balance realization method can interpret as an assignment problem. The well known Hungarian algorithm can utilize and solve this problem. Using this approach the undesired pairs can also avoided by some modification on the algorithm. The presented method is a systematic method which can obtain the optimal controllable and observable pairs online.

**References**