Analysis of Disturbance Observer Based Control for Nonlinear Systems under Disturbances with Bounded Variation

Wen-Hua CHEN and Lei GUO*  
Department of Aeronautical & Automotive Engineering  
Loughborough University, LE11 3TU, UK
Email:w.chen@lboro.ac.uk
* Now with Research Institute of Automation
Southeast University, Nanjing, 210096, China

Abstract—Control of nonlinear systems with unknown disturbances is considered in this paper using disturbance observer based control (DOBC) approach. The disturbances concerned are supposed to be time-varying but with bounded variation. For a nonlinear system with well-defined relative degree from the disturbances to the output, a nonlinear disturbance observer is presented to estimate the unknown disturbances and then is integrated with a conventional controller using disturbance observer based techniques. Stability and performance analysis of the composite closed-loop system is provided using Lyapunov theory and input-to-state stable concept.

I. INTRODUCTION

Disturbances widely exist in engineering. This is one of the main reasons why feedback is so prevalence. Control of nonlinear systems under disturbance has been an active research area in the last decades and several elegant approaches have been presented. Basically, control methods addressing disturbance attenuation can be classified according to the assumptions made on disturbances. The first approach is to model disturbances by a stochastic process, which leads to nonlinear stochastic control theory [1]. The second approach is nonlinear $H_{\infty}$ control [16] where the energy of disturbances is assumed to be bounded. The third approach is to deal with disturbances generated by a neutral stable exogenous system, namely, nonlinear output regulator theory [13]. These directions have been extensively investigated and many elaborate results have been developed. However, in general, design of controllers using these approaches involves the solution of a set of nonlinear partial differential equations (PDEs) or nonlinear partial differential inequalities (PDIs). Although significant progress has been achieved in this area and research is still going on, solution of those PDEs or PDIs, in general, can only be obtained for special cases or approximately. In addition to the above general approaches, there are also methods to deal with specific disturbances, for example, harmonics [4]. By sufficiently using the information of the specific disturbances, elegant results can be achieved.

This paper addresses a class of disturbances with bounded variation. This class of disturbances has been considered by [3]. Compared with other modelling of disturbances where the information of the distribution, the stochastic properties or the model of exogenous systems is required, less information is required in this modelling. Actually, the only requirement is that the variation of the disturbance is bounded. Many engineering disturbances can be approximately represented by this kind of disturbance after its high frequency part is filtered by properly designed filters in the loop or the inertia of systems.

A disturbance observer based control (DOBC) approach is adopted in this paper. In this approach the influence of disturbances is estimated by a disturbance observer and then compensated for based on the estimate. DOBC has its roots in many mechatronics applications in the last two decades [14], [9], [10], in particular for linear systems. Recently attempt has been made to establish theoretic justification of these DOBC applications and extend DOBC from linear systems to nonlinear systems. In the current DOBC framework [7], [6], similar to nonlinear regulation theory [13], it is assumed that the disturbances are generated by an exogenous system with a known model. However, the stability results established under this assumption may be invalid for engineering applications since it is quite difficult to model the disturbances as an output of an exogenous system for some engineering problems. Although simulation and experiment results have demonstrated that the DOBC may work quite well even though this assumption is not satisfied, it lacks theoretic justification of these applications [7], [5]. It is obvious that this obstructs the further development and applications of DOBC. To this end, this paper intends to overcome this restriction by only assuming that the disturbances have bounded variation. It is shown that, under a properly designed nonlinear disturbance observer and a proper feedback, the tracking error or regulation error caused by disturbances can be suppressed to an arbitrarily specified arrange.

II. PROBLEM FORMULATION

Consider a nonlinear system described as

$$\dot{x} = f(x) + g_1(x)u + g_2(x)w$$

(1)
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \) and \( w \in \mathbb{R} \) are the state vector, input and disturbance, respectively.

It is assumed that the disturbance \( w \) is unknown, time-varying but with bounded variation. That is,

\[
|\dot{w}| \leq \overline{\omega}_d,
\]

where \( \overline{\omega}_d \) is an unknown positive constant. Many engineering disturbances consist of both high frequency and low frequency parts. After they are filtered by a properly designed filter in the loop or the inertia of dynamic systems, they can be approximated by (2). Hence (2) represents a large variety of disturbances in engineering.

The objective of this paper is to design a nonlinear controller using the DOBC concept such that the influence of the disturbance satisfying (2) can be attenuated to a specified level. Within this framework, a nonlinear disturbance observer is required to estimate the influence of the disturbance and then it is integrated with a conventional controller. To establish our theoretic results, the following assumption is imposed in this paper.

**Assumption 1:** There exists a smooth function \( h(x) \) such that the relative degree from \( w \) to \( h(x) \), \( \rho \), is uniform well-defined for all \( x \) and \( t \).

**Remark 1:** An obvious choice of \( h(x) \) is the output of the dynamic system (1). However, if the relative degree from the disturbance to the output is not uniform well defined, a smooth function \( h(x) \) should be found such that Assumption 1 is satisfied.

### III. Nonlinear Observer for Uncertain Disturbances

In this paper, a nonlinear disturbance observer to estimate the disturbance \( w \) of system (1) is proposed as

\[
\begin{align*}
\dot{z} &= -l(x)g_2(x)[z + p(x)] - l(x)[f(x) + g_1(x)w] \\
\dot{w} &= z + p(x)
\end{align*}
\]

where \( p(x) \) is a nonlinear function to be designed and relates to the observer gain function \( l(x) \) as

\[
l(x) = \frac{\partial p(x)}{\partial x}.
\]

Let \( n_0 \) denotes \( \min_x |L_{g_2}L_{f}^{\rho-1}h(x)| \). Assumption 1 implies that \( L_{g_2}L_{f}^{\rho-1}h(x) \neq 0 \) for all \( x \). Hence \( n_0 \) is a positive scalar.

The performance of the nonlinear disturbance observer (3) is stated in Theorem 1 [8].

**Theorem 1:** Consider system (1) satisfying Assumption 1 and subject to the disturbance (2). When the nonlinear observer gain \( l(x) \) and the nonlinear function \( p(x) \) are chosen as

\[
l(x) = p_0 L_{f}^{\rho-1}h(x),
\]

and

\[
p(x) = p_0 L_{f}^{\rho-1}h(x)
\]

respectively, there always exists a constant \( p_0 \) such that the observer error dynamics is input-bounded-output-bounded stable.

Furthermore, the steady state estimation error yielded by the disturbance observer (3) is bounded by

\[
\frac{\overline{\omega}_d}{p_0 n_0}
\]

where \( \overline{\omega}_d \) denotes the maximum variation rate of the disturbance.

**Remark 2:** It should be noticed that \( p_0 \) should be chosen to be positive when \( n(x) > 0 \) and to be negative when \( n(x) < 0 \). The convergence of the disturbance estimate can be specified by adjusting the design parameter \( p_0 \).

### IV. Stability Analysis of DOBC

In the DOBC scheme, the control consists of two parts, given by

\[
u(t) = \alpha(x) + \beta(x)\hat{w}
\]

where \( \alpha(x) \) is a conventional control law designed under the assumption that there are no disturbances or the disturbances are known, and \( \hat{w} \) is the estimate yielded by the disturbance observer (3).

Substitution of the above control law into system (1) gives

\[
\dot{x} = f(x) + g_1(x)\alpha(x) + (g_1(x)\beta(x) + g_2(x))w - g_1(x)\beta(x)e
\]

where the observer error is governed by the nonlinear dynamics

\[
\dot{e} = \hat{w} - l(x)g_2(x)e
\]

The overall closed-loop dynamics consist of the controlled system dynamics (9) and the observer error dynamics (10). The structure of the overall closed-loop system is shown in Figure 1.

Stability of system (1) under DOBC (8) is stated in Theorem 2 and its proof is given in Appendix A.

**Theorem 2:** There exists a DOBC law (8) such that the closed-loop system (9) is input-to-state stable in the sense that when the disturbance \( w \) and its derivative are bounded, the system state and the observer error are bounded if the control law \( u = \alpha(x) \) renders the closed-loop system input-to-state stable and the disturbance observer is given by (3) and satisfies the conditions in Theorem 1.

Theorem 2 establishes the stability of the closed-loop system under the DOBC scheme. \( \beta(x) \) in the the control law
can be constructed following the procedure in Reference [12, p.27-28]. However, for a class of nonlinear systems where the disturbance satisfies so-called matching condition [2], more explicit results will be established in the following. A disturbance satisfies matching condition if it can be directly reached by system inputs. This kind of nonlinear system has been investigated by many nonlinear control methods. In the following it is assumed that there exists a nonlinear function $g_0(x)$ such that

$$g_1(x)g_0(x) = g_2(x) \quad (11)$$

This condition can also be extended to the nonlinear systems that can be transformed to satisfy it by a diffeomorphism [11].

When the nonlinear system (1) satisfies the matching condition (11) and the DOBC law (8) is applied, it follows from (9) that the controlled system dynamics are described by

$$\begin{align*}
\dot{x} &= f(x) + g_1(x)\alpha(x) \\
&\quad + (g_1(x)\beta(x) + g_2(x))w - g_1(x)\beta(x)e \\
&\quad = f(x) + g_1(x)\alpha(x) + g_2(x)e \\
\end{align*} \quad (12)$$

The structure of the overall closed-loop system consisting of the observer error dynamics (10) and the above controlled dynamics, (12), is shown in Figure 2. The stability of the DOBC for this class of nonlinear systems is established in Theorem 3 and its proof can be found in Appendix B.

**Theorem 3**: Consider a nonlinear system satisfying Assumption 1 and matching condition (11). Suppose that the control law $u = \alpha(x)$ renders the closed-loop system input-to-state stable. Then the closed-loop system under the DOBC (8), where $\dot{w}$ is given by the nonlinear disturbance observer (3) with nonlinear gain (5),(6) and a properly chosen $p_0$, and $\beta(x)$ is chosen as $-g_0(x)$, is input-to-state stable in the sense that when the variation rate of the disturbance is bounded, the observer error and the system state are bounded.

**Remark 3**: Theorem 3 points out that when the matching condition is satisfied, the bounds of the state and the observer error under the DOBC scheme only depend on the variation rate of the disturbances, which is different from Theorem 2 where the bounds of the state and the observer error depend on both the amplitude of the disturbance and its derivative. In particular, when the disturbance is constant in a period of time, zero steady state error can be achieved.

V. SIMULATION EXAMPLE

An electro-mechanical system with a nonlinear spring can be modelled as a controlled Duffing’s equation, described by

$$m\ddot{x} + c\dot{x} + f_1x + f_3x^3 = k_1u + b \quad (13)$$

where $x$, $u$ and $b$ are displacement, controller force and disturbance respectively. $m$, $c$ and $k_1$ are the mass, damping and the torque constant respectively. The characteristics of the nonlinear spring are represented by the parameters $f_1$ and $f_3$. $b$ is the external disturbance and it is unknown load in this case. The output is the displacement $y = x$.

For this simple nonlinear systems, a number of nonlinear control methods can be applied and the feedback linearisation technique is adopted in this paper which is given by

$$\begin{align*}
u(t) &= m(k_1(y_d - y) - k_0\dot{x})/k_t + \\
&\quad + (\dot{x} + f_1y + f_3y^3)/k_t \\
&\quad \text{with properly chosen control gain } k_0 \text{ and } k_1, \text{ good tracking performance can be achieved.}
\end{align*} \quad (14)$$

However the feedback linearisation control law designed above may have quite poor disturbance attenuation ability. Consider that unknown variable load is applied on this electro-mechanical system. For example, when the load time history given in Fig. 3 is applied, the position error under the design nonlinear control law is shown in Figure 2 by the dashed line. The parameters for the simulation has the following values: $m = 1 \text{ kg, } c = 5 \text{ Ns/m, } f_1 = 100 \text{ N/m, } f_3 = 500,000 \text{ N/m}^3, k_1=1\text{N/v, } k_0 = 16\text{Nm/s, } k_1 = 99\text{N/m}$. With the increase of the load, the position error significantly increases.

Then a nonlinear disturbance observer is designed and integrated with the above designed controller. As shown by the solid line in Fig 4, the position error is significantly reduced. Moreover the magnitude of the position error does not depend on the magnitude of the load but the variation rate of the load. For the period of from 10 second to 20 second, the load is quite large but since it remains a constant value, the position error gradually reduces to zero. This confirms the theoretic result presented in Theorem 3. To further investigate the performance of the proposed approach, random noise is added in the unknown load. It is found that the controller maintains good performance.

VI. CONCLUSION

A DOBC approach to the control problem of nonlinear systems with a large variety of disturbances was presented in this paper. The uncertain disturbances could be time-varying and no explicit modelling of the disturbances is required. This work significantly extends the applicability of the DOBC approaches for nonlinear systems and provides theoretic justification for some DOBC applications where the assumption that the disturbance is generated by a known model is not satisfied.

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Fig. 3. Time history of unknown load

Fig. 4. Position error comparison with and without disturbance observer

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APPENDIX A: PROOF OF THEOREM 2

When the control law $u = \alpha(x)$ is applied to system (1), the closed-loop system is given by

$$\dot{x} = f(x) + g_1(x)\alpha(x) + g_2(x)w$$  \hspace{1cm} (15)

Condition (1) in Theorem 2 implies that there exists a Lyapunov function $V(x)$ such that

$$\frac{\partial V}{\partial x}(f(x) + g_1(x)\alpha(x) + g_2(x)w) \leq -\gamma_1(||x||)$$  \hspace{1cm} (16)

when $||x|| \geq c_1(||w||)$ where $\gamma_1(\cdot)$ and $c_1(\cdot)$ are a class $K_\infty$ and $K$ function respectively [12, p.7].

When the disturbance is estimated by the disturbance observer (3) and the DOBC law (8) is applied, as shown in Figure 1, the closed-loop system is described by (9). It can be shown that if $||x|| \geq c_1(||w||)$, the derivative of the Lyapunov function $V(x)$ along the system’s trajectory satisfies

$$\phi = \frac{\partial V(x)}{\partial x}(f(x) + g_1(x)\alpha(x) + g_2(x)w - g_1(x)\beta(x)e)$$

$$\leq -\gamma_1(||x||) + \frac{\partial V(x)}{\partial x}g_1(x)\beta(x)\dot{w}$$  \hspace{1cm} (17)

Similar to the proof in [12, p.27], it can be concluded that there exists a $\beta(x)$ such that

$$\phi \leq -\frac{1}{2}\gamma_1(||x||)$$  \hspace{1cm} (18)

when $||x|| \geq c_2(||\dot{w}||)$ and $||x|| \geq c_1(||w||)$ where $c_2(\cdot)$ is a class $K$ function. It follows from (10) that these two conditions are implied by

$$||x|| \geq c_3\left(\frac{e}{w}\right)$$  \hspace{1cm} (19)

where $c_3(\cdot)$ is also a class $K$ function.

When an Lyapunov function candidate for the overall closed-loop system is chosen as

$$V_o(x, e) = V(x) + \frac{1}{2}e^2,$$  \hspace{1cm} (20)

its derivative along the trajectory of the closed-loop system in Figure 1 is yielded by

$$\dot{V}_o(x, e) = \frac{\partial V}{\partial x}\dot{x} + e_1\dot{e}_1$$

$$= \phi - p_0n_0e_1 + e_1\dot{w}$$  \hspace{1cm} (21)
It follows from (18) that
\[ \dot{V}_a(x,e) \leq -\frac{1}{2} \gamma_1(||x||) - p_0 n_0 e_1 + e_1 \dot{w} \]  
if condition (19) is met.
Furthermore, it can be shown that
\[ \dot{V}_a(x,e) \leq -\frac{1}{2} \gamma_1(||x||) \]  
if
\[ \|e\| \geq \left| \frac{\dot{w}}{p_0 n_0} \right| \]  
Combining condition (24) with (19) obtains
\[ \|x\| \geq c_3 \left( \begin{array}{c} e \\ w \end{array} \right) \]  
\[ \geq c_4 \left( \begin{array}{c} \dot{w} \\ w \end{array} \right) \]  
(25)
where \( c_4(\cdot) \) is also a class \( K \) function. Hence it has been proved that (23) is met if \((x,e)\) satisfies condition (24) and (25). This implies that the observer error and the system state of the closed-loop system under bounded \( \dot{w} \) and \( w \) in Figure 1 are bounded. Therefore the closed-loop system under the DOBC is input-to-state stable.

**APPENDIX B: PROOF OF THEOREM 3**

Suppose that there exists \( u = \alpha(x) \) such that system (1) is input-to-state stable. Similar to the proof of Theorem 2, this implies that there exists an Lyapunov function \( V(x) \) such that
\[ \dot{V}(x) = \frac{\partial V(x)}{\partial x}(f(x) + g_1(x)\alpha(x)) + \frac{\partial V(x)}{\partial x} g_2(x)w \leq -\gamma_1(||x||) + \frac{\partial V(x)}{\partial x} g_2(x)w \]  
(26)
Furthermore there exist a class \( K_\infty \) function \( \gamma_2(\cdot) \) and a class \( K \) function \( c_1(\cdot) \) such that
\[ \dot{V}(x) \leq -\gamma_2(||x||) \]  
(27)
if
\[ \|x\| \geq c_1(||w||). \]  
(28)
Define \( V_o(x,e) = V(x) + \frac{1}{2} x^2 \) as an Lyapunov candidate for the overall closed-loop system in Figure 2. Differentiation of the Lyapunov function along the trajectory of the overall closed-loop system consisting of the observer error dynamics (10) and the controlled system dynamics (12) yields
\[ \dot{V}_a(x,e) = \frac{\partial V(x)}{\partial x} \dot{x} + e \dot{e} \]  
\[ = \frac{\partial V(x)}{\partial x}(f(x) + g_1(x)\alpha(x)) \]  
\[ + \frac{\partial V(x)}{\partial x} g_2(x)e + e(-p_0 n(x)e + \dot{w}) \]  
(29)
Similar to the proof of Theorem 1, without loss of generality, it is assumed that \( n(x) > 0 \) for all \( x \) and \( t \). Using the results in (26), (27) and (28), one can conclude that
\[ \dot{V}_a(x,e) \leq -\gamma_1(||x||) + \frac{\partial V(x)}{\partial x} g_2(x)e + e(-p_0 n(x)e + \dot{w}) \]  
\[ \leq -\gamma_2(||x||) - p_0 n_0 e^2 + e\dot{w} \]  
\[ \leq -\gamma_2(||x||) - p_0 n_0 e^2 + ew \]  
(30)