Dynamic sliding mode control of nonlinear systems

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Abstract

Dynamic sliding mode control and higher order sliding mode are studied in this paper. Dynamic sliding mode control adds additional dynamics, which can be considered as compensators. The sliding system with compensators is an augmented system. These compensators (extra dynamics) are designed for achieving and/or improving the system stability, hence obtaining desired system behaviour and performance. Higher order sliding mode control and dynamic sliding mode control yield more accuracy and also reduce and/or remove the chattering resulting from the high frequency switching of the control. It is proved that certain $J$-trajectories reach a sliding mode in a finite time. A sliding mode control differentiator is also considered.

1 Introduction

Sliding Mode Control (SMC) has widely been extended to new techniques, such as Higher Order Sliding Mode Control (HOSMC) [1,3,4] and Dynamic Sliding Mode Control (DSMC) [5,7-11]. These techniques retain the main advantages of SMC and also yield more accuracy. These techniques can also be applied for designing observers to differentiate signals achieving robustness in the presence of noise.

SMC is a high frequency switching control signal to enforce the system trajectories onto a surface, the so-called sliding surface (or hyperplane), after a finite time and remain within the vicinity of the sliding surface towards the equilibrium point, thereafter [10]. The sliding surface is designed to achieve desired specifications. SMC is robust with respect to matched internal and external disturbances. However, undesired chattering produced by the high frequency switching of the control may be considered a problem for implementing the sliding mode control methods for some real applications. Methods have been presented to reduce the chattering, for instance the continuous approximation technique [10]. Another way is to design HOSMC control. A drawback of continuous approximation methods is the reduction of the accuracy of the system and the sliding mode stability. SMC techniques are applicable to any minimum phase system with relative degree less than the system order. There are some SMC techniques for stabilisation of non-minimum phase systems including use of DSMC, for stabilising the internal dynamics when the output tracking error tends asymptotically to zero in the sliding mode [8,9].

DSMC has received attention in recent years [5,8-11]. Introducing extra dynamics into a sliding surface helps to solve many difficulties in practice, such as flight control design and time scale separation of control loops in a multi-loop system [7,8]; replacement of a state observer to achieve stability under incomplete information about actuator dynamics [6]; and, even accommodation of unmatched disturbances extending the system state space into the exogenous states of an unknown signal modelled by linear dynamics. This method can be applied to the non-minimum phase tracking problem, e.g. stabilization in a dynamic sliding manifold of tracking error dynamics together with unstable internal dynamics, plus unmatched input exogenous dynamics with insufficient information about states of this composite system [8]. DSM provides stability to the internal states and asymptotic stability to the states of the tracking error dynamics.

Young and Özgüner [11] and Koshkouei and Zinober [5] have designed compensators using the optimal control and realisation methods for linear systems in the sliding mode. The sliding system with a compensator (extra dynamics) is an augmented system with a higher order system compared with the original system. However, the designed compensators may not only improve the stability of the sliding system but also yield desired performance and characteristics. In this paper, a nonlinear compensator for linear and nonlinear systems is designed.

The paper is organized as follow: HOSMC is presented in Section 2 and the dynamic sliding mode is considered in Section 3. Also, in this section the properties of DSMC and the reaching time are studied. The defined sliding dynamics moves on the appropriate sliding surface faster than the system sliding mode. Section 4 deals with SMC differentiators. In Section 5 an example is presented to illustrate the theory. Conclusions are presented in Section 6.
2 Higher order sliding mode control

HOSMC is a way to improve the accuracy of the sliding mode and remove chattering.

Consider a system of the form
\[ \dot{x} = A(x,t) + B(x,t)u \]  
(1)
where \( x \in \mathbb{R}^n \) is the state and the scalar control \( u \in \mathbb{R} \). \( A(x,t) \) and \( B(x,t) \) are smooth vector fields. Define the sliding function as \( \sigma = \sigma(x,t) \). Suppose the system’s relative degree \( r \) with respect to \( \sigma = \sigma(x,t) \) is constant and known. Then the system (1) is transferred to new coordinates \( \sigma^{(r)} = f(t, \sigma, \sigma, ..., \sigma^{(r-1)}, \zeta) + g(t, \sigma, \sigma, ..., \sigma^{(r-1)}, \zeta)u \)

\[ \zeta = \phi(t, \sigma, \sigma, ..., \sigma^{(r-1)}, \zeta) + \psi(t, \sigma, \sigma, ..., \sigma^{(r-1)}, \zeta) \]  
(2)
where \( \zeta = (\zeta_1, ..., \zeta_{n-r+1}) \) [2]. The \( r \)th-order sliding mode exists if there is a control \( u \) such that the zero dynamic equations \( \sigma = \dot{\sigma} = \cdots = \sigma^{(r)} = 0 \) are satisfied. The control during the sliding mode, the so-called equivalent control, is

\[ u_{eq} = -f(\zeta)/g(t, \zeta) \]  
(3)
[3]. Substituting (3) into (1) yields \( r \)-sliding mode motion, which is the zero dynamics of the system (1). Moreover, the system is stable if the system is minimum phase. The results of second order SMC are interesting; in particular it can be successfully applied to real problems. Various methods for designing 2-order SMC and applications have been presented [1,3].

Attention is now focused on the second order sliding mode systems. Suppose the system (1) has the relative degree one with respect to \( \sigma = \sigma(t,x) \). Then the system (2) is

\[ \ddot{\sigma} = f(t, \sigma, \dot{\sigma}) + g(t, \sigma, \dot{\sigma})u \]  
(4)
The additional dynamics yields greater accuracy of the control design and suitable response behaviour and performance. These dynamics act as a compensator for the system. The resulting controller has the features of traditional SMC such as insensitivity to matched disturbances and nonlinearities and a classical dynamics compensator with accommodation of unmatched disturbances. For simplicity, consider a system with the following sliding dynamics

\[ \ddot{\sigma} = f(t, \sigma) + u \]  
(5)
For the existence of sliding mode control it is sufficient to show that the trajectories reach the sliding surface \( \sigma = 0 \) in a finite time and remain on it thereafter.

3 Dynamic sliding mode control

The system in the sliding mode may need some additional dynamics to improve the system stability and the sliding mode stability as well as obtaining the desired system response and behaviour.

It may also require a controller to be designed such that the output of an uncertain SISO dynamic system tracks some real-time measured signal.

When the output is measurable, the convergence time is required to be finite so that the tracking is robust with respect to measurement errors and exact in their absence. In order to solve such a problem some additional assumptions may still be needed.

3.1 Dynamic sliding surface

A dynamic sliding function \( \sigma \) is defined as a linear operator, which has a realization as a linear time-invariant dynamic system

\[ \dot{\chi} = F\chi + G_1e + G_2x \]
\[ \sigma = C\chi + He + Kx \]

\( x \) is the state of the original system, \( e = x - x_d \) is the error variable with \( x_d \) as a desired signal. \( \chi \) is a state resulting from realization of the operator \( \sigma \). \( F, G_1, G_2, C, H, K \) and \( K \) are matrices which show the relationship between the states with compatible dimensions. For minimum-phase systems \( G_2 = 0 \) and \( K = 0 \) whilst for non-minimum-phase systems they are nonzero [9].

3.2 Dynamic sliding mode (DSM)

The DSM in the dynamically extended state space is defined as \( \sigma = 0 \). The system output tracks the desired value if \( \sigma = 0 \) so that matched uncertainties and disturbance do not affect the tracking. In fact, for a sliding mode linear system (i.e. the system during the sliding mode, \( \sigma = 0 \)) the desired system response and performance can be achieved by selecting a set of prespecified eigenvalues.

Nonlinear sliding mode dynamics are now introduced. By defining such dynamics, two different sliding surfaces are obtained. However, there is a close relationship between them. A sliding mode control is designed using the new component. Define the \( J \) nonlinear dynamic sliding mode

\[ \dot{\chi} = a |\sigma|^{0.5} \text{sgn}(\sigma) - b |\chi + \sigma|^{0.5} \text{sgn}(\chi + \sigma) \]
\[ J = \chi + \sigma \]  
(6)
with \( a > 0, b > 0 \) and \( a \neq b \) [7,8]. \( \chi \) is an error variable of two sliding mode variables \( \sigma \) and \( J \). Using the second order sliding mode, a sliding mode control \( u \) is designed such that \( \sigma = \dot{\sigma} = 0 \). The problem is to find a suitable sliding mode control to guarantee a \( J \)-sliding mode (or \( \sigma \) -sliding mode). The following theorem guarantees the existence of the sliding mode with sliding mode control \( u = -\rho \text{sgn}(J) \) where \( \rho \) is a suitably large positive real number. In many practical
problems, one needs to know the relationship between the $J$- and $\sigma$-dynamics. In fact, it is desired that the $J$-sliding mode reaches the sliding surface $J = 0$ faster than $\sigma$-dynamics. In this case, \(\lim_{t \to \infty} J = 0\), where \(t_J\) is the reaching time to the sliding surface $J = 0$. The following theorem yields this relationship.

**Theorem 1:** Consider the sliding dynamics (5) and (6). Let \(a > 0\), \(b > 0\) and \(a \neq b\). Then the following statements are implied.

(i) The $J$-dynamics sliding mode exists.

(ii) The $J$-sliding mode occurs if and only if the $\sigma$-sliding mode exists.

(iii) The $J$-dynamics reaches and remains on the sliding surface $J = 0$ if and only if

\[ |J(0)| \leq \frac{(b^2)}{a} |\sigma(0)| \]

**Proof:**

(i) From (5) and (6), one can obtain

\[ \ddot{J} + \frac{1}{2} \left( \frac{b}{|J|^{0.5}} - \frac{a}{|\sigma|^{0.5}} \right) \dot{J} = \Phi(\cdot) + u \quad (7) \]

where \(\Phi(\cdot) = -\left( \frac{a^2}{2} \right) |\sigma| + \frac{ab}{2} |J|^{0.5} \sgn(J) + f(t, \sigma)\)

Assume that \(|\Phi(\cdot)| \leq L\). Consider the sliding mode control \(u = -\rho \sgn(J)\) with \(\rho > L\). Then (7) yields

\[ \dot{J} = -\frac{1}{2} \left( \frac{b}{|J|^{0.5}} - \frac{a}{|\sigma|^{0.5}} \right) \dot{J} + \Phi(\cdot) - \rho \sgn(J) \]

Let \(R = \rho - \Phi(\cdot) \sgn(J)\). Then

\[ R \in [\rho - L, \rho + L] \sgn(J) \]

and substituting into (8) implies

\[ \ddot{J} = \frac{1}{2} \left( \frac{b}{|J|^{0.5}} - \frac{a}{|\sigma|^{0.5}} \right) \dot{J} - R \sgn(J) \]

In the Filippov sense,

\[ \dot{J} = -\frac{1}{2} \left( \frac{b}{|J|^{0.5}} - \frac{a}{|\sigma|^{0.5}} \right) J - [\rho - L, \rho + L] \sgn(J) \]

where the right-hand side is a differential inclusion \(\dot{J}\). Suppose \(a \mid \sigma(0) \mid^{0.5} \neq b \mid J(0) \mid^{0.5}\). Consider the trajectory \(\Gamma_i\)

\[ \begin{cases} \rho - L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) > 0 \\ \rho + L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) < 0 \end{cases} \]

and the trajectory \(\Gamma_2\)

\[ \begin{cases} \rho + L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) > 0 \\ \rho - L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) < 0 \end{cases} \]

Any trajectory $\Gamma$ between the two trajectories \(\Gamma_1 \) and \(\Gamma_2\), crosses the $J$-axis (i.e. when \(b \mid \sigma(0) \mid^{0.5} = a \mid J(0) \mid^{0.5}\)) at \(J_1, J_2, \ldots\) so that for any \(i \geq 0\), \(|J_{i+1}| \leq |J_i|\). Therefore,

\[ \left\{ \left| J_i \right| \right\}_{i=1}^{\infty} \text{ is a nonnegative decreasing sequence. So } \lim_{t \to \infty} J = 0 \text{ and the second-order } J \text{-sliding mode occurs.} \]

(ii) From (6)

\[ \check{\sigma} + a \mid \sigma \mid^{0.5} \sgn(\sigma) = \check{J} + b \mid J \mid^{0.5} \sgn(J) \]

Therefore, \(\check{\sigma} + a \mid \sigma \mid^{0.5} \sgn(\sigma) \leq 0\) if and only if \(\check{J} + b \mid J \mid^{0.5} \sgn(J) \leq 0\). This is equivalent to \(\check{\sigma} \leq -a \mid \sigma \mid^{1.5}\) if and only if \(\check{J} \leq -b \mid J \mid^{1.5}\). So \(\check{\sigma} \sigma \leq 0\) if and only if \(\check{J} J \leq 0\).

(iii) Assume that \(\check{\sigma} + a \mid \sigma \mid^{0.5} \sgn(\sigma) \leq 0\) then \(\check{\sigma} \to 0\) and \(\sigma \to 0\) in a finite time. So

\[ t_\sigma = \frac{|\sigma(0)|^{0.5}}{2a} \]

Consequently, from \(\check{J} + b \mid J \mid^{0.5} \sgn(J) \leq 0\) one can see that \(\check{J} \to 0\) and \(J \to 0\) in a finite time and

\[ t_J = \frac{|J(0)|^{0.5}}{2b} \]

This completes the proof of (iii).

Furthermore, if the conditions

\[ \lim_{t \to \infty} f(t, 0, 0) = \lim_{t \to \infty} g(t, 0, 0) = 0 \]

are satisfied and \(\sigma \to 0\) and \(\check{\sigma} \to 0\), then (5) yields \(\lim_{t \to \infty} \check{\sigma} = 0\).

### 4 Estimation of differentiation using DSMC

A DSMC differentiator is presented using the theory in Section 3. A DSMC differentiator was introduced by Levant [4]. The Levant differentiator is as follows: Let \(y = x + \eta\) where \(\eta\) is a Gaussian noise and \(y\) is a measurable variable. An estimation of \(\hat{x}\) is required. An estimate of \(\hat{x}\) is

\[ \hat{x}(t) = a \mid e(t) \mid^{0.5} \sgn(e(t)) + b \int_0^t \sgn(e(s))ds \]

\(e(t) = y(t) - \hat{x}(t)\)

where \(\hat{x}\) is an estimate of \(x\). Select \(x_1 = \hat{x}\). It is possible to estimate a differentiator for \(x_1\) as

\[ \Gamma_1 \]

\[ \begin{cases} \rho - L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) > 0 \\ \rho + L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) < 0 \end{cases} \]

\[ \begin{cases} \rho + L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) > 0 \\ \rho - L & \text{if } (b \mid \sigma(0) \mid^{0.5} - a \mid J(0) \mid^{0.5})J(0) < 0 \end{cases} \]
\[ \dot{x}_i(t) = a |e_i(t)|^{0.5} \text{sgn}(e_i(t)) + b \int_0^t \text{sgn}(e_i(s)) ds \]

\[ e_i(t) = y_i(t) - \hat{x}_i(t) \]

where \( \hat{x}_i \) is an estimate of \( x_i \). In fact, (15) yields an estimation of the second order differentiation of \( x \). This process can be applied a finite number of times to obtain the desired higher order differentiation.

Shtessel [7,8] has introduced the following differentiator

\[ \dot{x} = a |e|^{0.5} \text{sgn}(e) - b |J|^{0.5} \text{sgn}(J) \]

\[ J = \dot{x} + e \]

\[ e = y - \dot{x} \]

\[ \dot{e} = -K \text{sgn}(J) \]

where \( \dot{x} \) is an estimate \( x \). Let \( y_1 = \dot{x} \). The new differentiator can be defined as

\[ \dot{x}_1 = a |e_1|^{0.5} \text{sgn}(e_1) - b |J_1|^{0.5} \text{sgn}(J_1) \]

\[ J_1 = \dot{x}_1 + e_1 \]

\[ e_1 = y_1 - \dot{y}_1 \]

\[ \dot{y}_1 = -K_1 \text{sgn}(J_1) \]

which can be considered as a filter for the estimation of \( \dot{x} \). In this way, a finite series of filters can be produced. Theorem 1 implies that for any \( i \geq 0 \), the \( e_i \) - and \( J_i \) - sliding mode trajectories converge to the sliding surfaces \( e_i = 0 \) and \( J_i = 0 \) in finite time and if an appropriate condition is satisfied, the \( J_i \)-dynamics converges faster than \( e_i \) -dynamics.

## 5 Example

Consider a system that can be converted to the sliding dynamics (5)

\[ \dot{\sigma} = f(t, \sigma) + u \]

where

\[ f(t, \sigma) = 2\sigma^2 - \sigma - 2 \sin(2t - 0.5) \]

and

\[ u = -K \text{sgn}(J) \]

This may arise from a system with tracking signal \(-2 \sin(2t - 0.5)\). Consider the nonlinear dynamic sliding mode (6)

\[ \dot{x} = a |\sigma|^{0.5} \text{sgn}(\sigma) - b |\dot{x} + \sigma|^{0.5} \text{sgn}(\dot{x} + \sigma) \]

\[ J = \dot{x} + \sigma \]

with \( a > 0 \), \( b > 0 \) and \( a \neq b \). According to Theorem 1 the sliding mode \( J \)- and \( \sigma \)-dynamics exist. Assume that \( a > 0 \), \( b > 0 \) \( a \neq b \) and \( |J(0)| \leq (\frac{b}{a})^3 |\sigma(0)| \). Then the \( J \)-dynamics trajectories reach the sliding surface \( J = 0 \) before the \( \sigma \)-dynamics trajectories hit the sliding surface \( \sigma = 0 \). For simulation select \( a = 1 \), \( b = 2 \), \( J(0) = 0.1 \), \( \sigma(0) = 1 \) and \( \sigma(0) = 0.5 \). The condition (iii) of Theorem 1 is satisfied

\[ |J(0)| = 0.1 \leq (\frac{b}{a})^3 |\sigma(0)| = 4 \]

Therefore, the \( J \)-dynamics is faster than \( \sigma \)-dynamics. The simulations also show this result. The reaching time of the \( J \)-sliding mode is less than 0.4 whilst the \( \sigma \)-sliding mode reaching time is larger than 1.7 (see Fig. 2). Fig. 2 also shows that \( \dot{\sigma} \) converges to 0 at finite time \( t_o \approx 2 \). Fig. 2 also illustrates, the behaviour of the sliding mode control and the sliding mode reaching time; thereafter the \( \sigma \)-trajectories remaining on the sliding surface \( \sigma = 0 \). Fig. 1 illustrates the behaviour of the nonlinear function \( f(t, \sigma) \) with respect to time, which shows that the nonlinear function \( f(t, \sigma) \) does not tend to 0 when \( t \to \infty \). However, since \( \dot{\sigma} = 0 \), for \( t > 0.4 \), \( f(t, \sigma) = -2 \sin(2t - 0.5) \). When \( a = b \), and the initial conditions is \( \sigma(0) = J(0) \), then the behaviour of the \( J \)-dynamics and \( \sigma \)-dynamics coincide and the sliding system is marginally stable (see Fig. 3). In Fig. 3, the lower plot of the second column, depicts the control action when \( a = b \) and \( \sigma(0) = J(0) \). The switching between two control values is not very fast in comparison with the case when the discontinuous control, \( u = -15 \text{sgn}(J) \) with \( a = 1 \), \( b = 2 \) is applied. See Fig. 2. In fact, the control is retained as a constant value, \( \pm K \), for a while and then switches to another value, \( \mp K \). This process repeatedly occurs. For example, if the control is \( K \) for a certain time, then it is switched to \( -K \) and after another certain period of time, the control is switched to \( K \) again. The control is a rectangular signal, which has a constant amplitude and a different width. Its width depends on the \( J \)-dynamics behaviour periods. The behaviour of the function \( f(t, \sigma) \) with these conditions is shown in Fig. 4, which completely different from Fig. 1. In this case

\[ f(t, \sigma) = 2\sigma^2 - \sigma - 2 \sin(2t - 0.5) \]

## 6 Conclusions

Dynamic and higher order sliding mode controls have been studied in this paper. DSMC is a technique for improving and/or achieving the system stability or desired behaviour, by designing compensators. This paper has presented some conditions for reaching trajectories to the appropriate sliding surfaces. The prediction of the behaviour of different sliding mode dynamics is important for designing a sliding mode control for achieving the sliding mode stability and furthermore, the system stability. Using the main theorem in this paper the DSMC differentiator has also been studied.
References


Fig. 1. The behaviour of the function $f(\sigma, t)$ for $a=1$ and $b=2$.

Fig. 2. The response of the sliding dynamics with discontinuous control $u = -15 \text{sign}(J)$ when $a=1$ and $b=2$.

Fig. 3. The response of the sliding dynamics with discontinuous control $u = -15 \text{sign}(J)$ when $a = b (= 2)$, and the initial conditions are $\sigma(0) = J(0)$.

Fig. 4. The behaviour of the function $f(\sigma, t)$ for $a = b$, and the initial conditions $\sigma(0) = J(0)$. 