GLOBAL STABILITY OF MULTIVARIABLE SYSTEMS WITH ACTUATOR SATURATION COMPENSATION

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Keywords: Actuator saturation, saturation compensation, global stability, nonlinear system.

Abstract

As most practical control systems are subject to actuator saturation, it is important that these systems remain stable after actuator saturation has occurred. In this paper, a general actuator saturation compensator for multivariable systems is proposed, which is constructed by solving the strictly positive real control problem. The design of compensator for systems with an integrator is also considered, and the performance of the proposed saturation compensator is illustrated by an example involving a system containing an integrator.

1 Introduction

Most practical control systems are subject to actuator saturation. When the actuator saturates, the performance of the controller can deteriorate giving rise to larger overshoots and longer settling time. In the worst case, the closed-loop system can be unstable. There are two approaches to tackle this problem. The first one is to design the controller with actuator saturation as a constraint [7-9]. The main drawback of this approach is that the response of the closed-loop system may be sluggish [2] when there is no actuator saturation. In the second approach, techniques developed are widely known as the anti-windup methods [1,4,5]. Since these schemes are often devised heuristically, it is difficult to obtain analytical results on the performance and the stability of the compensated system. The unified actuator saturation compensator proposed in [2,3], includes most existing anti-windup or bumpless transfer compensation schemes, such as the conditioning technique [5], and the observer-based techniques [1] as special cases. Based on this framework, it is shown that there exist compensators, such that a single-input-single-output (SISO) system with up to one integrator can be stabilized globally. However, the compensator design method proposed in [5] cannot be readily extended to Multi-Input-Multi-Output (MIMO) systems. It is proposed in this paper to transform the compensator design problem to a strictly positive real control problem. The MIMO compensator is then obtained from the solution of the resulting linear matrix inequality (LMI). The design of compensator for systems with an integrator is also considered, and the performance of the proposed compensator is illustrated by an example involving a system containing an integrator.

2 A dynamic actuator saturation compensator

Consider a linear MIMO system $P$ described by the minimal state-space realization,

$$
P: \begin{align*}
\dot{x}_p(t) &= Ax_p(t) + Bu(t) \\
y(t) &= Cx_p(t)
\end{align*}
$$

where $u(t) \in \mathbb{R}^m$ is the actuator output, $y(t) \in \mathbb{R}^p$ is the system output, $x_p(t) \in \mathbb{R}^n$ is the state system, $A \in \mathbb{R}^{n \times n}$ is the state matrix, $B \in \mathbb{R}^{n \times m}$ is the input matrix, and $C \in \mathbb{R}^{p \times n}$ is the output matrix. Denote a real matrix with $m$ rows and $n$ columns by $R^{m \times n}$, a real vector with dimension $n$ by $R^n$, and the transpose of matrix $M$ by $M^T$. Assume all the eigenvalues of $A$ are in the closed left-half of the s-plane. From (1), the transfer function matrix of the system is,

$$G(s) = C(sI - A)^{-1}B \quad (2)$$

Assume that the linear controller $K$ is designed such that the linear closed-loop system is asymptotically stable.

$$K : \begin{align*}
\dot{x}_d(t) &= Fx_d(t) + G_1w(t) - N_2y(t) \\
v(t) &= Hx_d(t) + G_2w(t) - N_3y(t)
\end{align*} \quad (3)$$

where $w(t) \in \mathbb{R}^m$ is the reference input, $v(t) \in \mathbb{R}^p$ is the controller output, $x_d(t) \in \mathbb{R}^n$ is the controller state, and $F \in \mathbb{R}^{n \times n}$, $G_1 \in \mathbb{R}^{n \times p}$, $N_2 \in \mathbb{R}^{p \times n}$, $G_2 \in \mathbb{R}^{n \times p}$, $N_3 \in \mathbb{R}^{p \times n}$ are constant matrices. It is assumed that all the eigenvalues of $F$ are in the closed left-half of the s-plane. The transfer function matrix of the controller (3) is

$$v(s) = \frac{1}{T(s)}w(s) - S(s)y(s) \quad (4)$$

where

$$T(s) = H(sI - F)^{-1}G_1 + G_2$$

$$S(s) = H(sI - F)^{-1}N_1 + N_2$$

The actuator output with known upper and lower bounds: $u_i^+$ and $u_i^-$ for $i = 1, 2, \cdots, m$, is

$$u(t) = sat[v(t)] = [sat[v_1(t)] \ sat[v_2(t)] \ sat[v_m(t)]] \quad (5)$$

where

$$sat[v_i(t)] = \begin{cases}
    u_i^+, & v_i(t) > u_i^+ \\
    v_i(t), & u_i^- \leq v_i(t) \leq u_i^+ \\
    u_i^-, & v_i(t) < u_i^-
\end{cases}$$

The closed-loop system with actuator saturation is shown in Fig. 1. From (5), the actuator output $u(t)$ is different from the controller output $v(t)$ when the actuator saturates,
which can lead to performance degradation, or even instability. Following [2,3], $\eta(t)$ is added to $v(t)$ to compensate for actuator saturation,

$$v(t) = Hx(t) + G_2w(t) - N_2\eta(t) + \eta(t)$$

and

$$\dot{x}_u(t) = A_u\eta_x(t) + B_\eta \delta(t)$$

$$\eta(t) = C_\eta x(t)$$

$$\delta(t) = u(t) - v(t)$$

$x_u(t) \in R^r$, $\eta(t) \in R^m$, $A_u \in R^{r \times r}$, $B_\eta \in R^r \times m$ and $C_\eta \in R^m \times r$ are matrices to be chosen by the user, such that the compensated system is asymptotically stable or some design criteria are satisfied. From (7), the transfer function matrix of the saturation compensator is

$$P(s) = C_\eta (sI - A_u)^{-1} B_\eta$$

From (6) to (9), the controller with actuator saturation compensation becomes

$$v(s) = T(s)(w(s) - S(s)) + P(s)\delta(s)$$

The closed-loop system with actuator saturation compensation is shown in Fig. 2. Its stability can be analyzed by first transforming it to an equivalent system, as shown in Fig. 3. As $v$ and $u$ in the compensated system and the equivalent system are identical, the compensated system has the same stability property as the equivalent system [3]. This result can be readily extended to MIMO system. From (1), (3) and (6) and (7), the adjoint system of the plant $P$, the controller $K$, and the saturation compensator is

$$\dot{x}_u(t) = A_u\eta_x(t) + B_\eta \delta(t)$$

$$\eta(t) = C_\eta x(t)$$

$$\delta(t) = u(t) - v(t)$$

where

$$A_u = \begin{bmatrix} A & 0 & 0 \\ -N_2C & 0 & 0 \\ B_\eta N_2C & -B_\eta H & A_\eta - B_\eta C_\eta \end{bmatrix}$$

From (2), (9) and (10), the controller output $v(s)$ can be rewritten as follows,

$$v(s) = -T(s)u(s) + T_d(s)w(s)$$

where

$$T_d(s) = (I + P(s))^2T(s)$$

$$G_d(s) = (I + P(s))^2(S(s)G(s) - P(s))$$

Equation (12) can be readily simplified to the SISO system given in [3]. The equivalent system of the compensated system is obtained from (12) such that $v$ and $u$ are identical to the compensated system. As $w$ is often set to zero in stability analysis, $T_d(s)$ can be ignored in the following analysis. The equivalent system (12) now reduces to

$$\Sigma_E : \begin{bmatrix} \dot{x}_u(t) = A_u x_u(t) + B_\eta \bar{\eta}(t) \\ v(t) = -N_2C \eta(t) + \bar{\eta}(t) \\ \eta(t) = C_\eta x(t) \end{bmatrix}$$

where $x_u(t) = [x_u^T(t) \ x_u^T(t) \ \bar{\eta}(t)]^T$. The results presented in [3] for SISO systems cannot be readily to MIMO systems. In the next section, a method is presented to design these based on the LMI is presented.

3 Global stability of the compensated system

A variable $x(t)$ is said to be asymptotically stable if $x(t)$ is bounded and approaches the origin as time tends to infinity for any finite initial state $x(0)$. The compensated system with actuator saturation is globally stable if all its states and output is asymptotically stable for all actuator limits. A proper rational transfer function matrix $Z(s) \in R^{n \times n}$ is said to be positive real if: (1) all elements of $Z(s)$ are analytic for $Re(s) > 0$, (2) any pure imaginary pole of any element of $Z(s)$ is a simple pole and the associated residue matrix of $Z(s)$ is positive semidefinite Hermitian, and (3) for all real $\omega$ for which $j\omega$ is not a pole of any element of $Z(s)$, the matrix $Z(j\omega) + Z^*(-j\omega)$ is positive semidefinite. The transfer function $Z(s)$ is strictly positive real (SPR) if $Z(s)\epsilon$ is positive real for some $\epsilon > 0$, and is extended strictly positive real (ESPR) if $Z(s)$ is analytic in $Re(s) \geq 0$ and satisfies $Z(j\omega) + Z^*(-j\omega) > 0$ for $\omega \in [0, \infty)$. From the definitions of SPR, ESPR and Lemma 10.1 in [11], $Z(s)$ must be Hurwitz if $Z(s)$ is SPR, and if $Z(\infty) + Z^*(\infty) > 0$, $Z(s)$ is SPR if and only if $Z(s)$ is ESPR, which is characterized by linear matrix inequality [10].

In this paper, it is assumed that the closed-loop system with no actuator saturation is asymptotically stable, i.e.,

$$A_w = \begin{bmatrix} A - B N_2C & BH \\ -N_2C & F \end{bmatrix}$$

is a stable matrix.

Rewriting (13) gives

$$\Sigma_E : \begin{bmatrix} \dot{x}_u(t) = A_u x_u(t) + B_\eta \bar{\eta}(t) \\ v(t) = C_\eta x(t) \end{bmatrix}$$

where $B_\eta = [-B' \ 0 \ -B'_\eta]$ and $C_\eta = [-N_2C \ H \ C_\eta]$. The output of the equivalent system $y_E$ is

$$y_E(t) = [C \ 0 \ 0]x(t)$$

The nonlinearity $\bar{\eta}(t) = -\eta(v) = -sat[v(t)]$ belongs to the conic sector $[K, I]$. When global stability is considered, $K$ approaches zero. If $A$ and $F$ are stable matrices, then from the circle criterion [6], $\Sigma_E$ is globally stable if there exists a compensator $P(s)$ such that $I + G_d(s)$ is SPM. However, the result does not necessarily hold if $A$ and/or $F$ are unstable, since $A_u$ is also unstable. The global stability of the compensated system is given below.
Theorem 1 The equivalent system $\Sigma_\varepsilon$ given by (13) is globally stable, if there exists a compensator given by (7) and (8), such that for an arbitrary small positive number $\varepsilon$, $(A_{\text{sat}}, B_3)$ is controllable, $(A_{\text{sat}}, B_0)$ is observable and $Z_\varepsilon(s)$ is strictly positive real, and $G_\varepsilon(s)$ is given by (12),

$$Z_\varepsilon(s) = (I + G_\varepsilon(s))(I + KG_\varepsilon(s))^{-1}$$

$$K = \text{diag}(k_1, k_2, \ldots, k_n) = \varepsilon I$$

Proof. From (13), $G_\varepsilon(s)$ has a state space realization,

$$G_\varepsilon(s) = \begin{bmatrix} A_{\text{sat}} - B_0KC_0 & B_0 \\ C_0 & 0 \end{bmatrix}$$

Similarly, $G_\varepsilon(s)(I + KG_\varepsilon(s))^{-1}$ and $(I + G_\varepsilon(s))(I + KG_\varepsilon(s))^{-1}$ can be written as,

$$Z_\varepsilon(s) = (I + G_\varepsilon(s))(I + KG_\varepsilon(s))^{-1} = \begin{bmatrix} A_{\text{sat}} - B_0KC_0 & B_0 \\ (I - K)C_0 & I \end{bmatrix}$$

Since $Z_\varepsilon(s)$ is SPR for arbitrary small positive $k_i = \varepsilon$, $Z_\varepsilon(s)$ is Hurwitz. It follows that $A_{\text{sat}} - B_0KC_0$ is stable, and from (17), $G_\varepsilon(s)(I + KG_\varepsilon(s))^{-1}$ is also Hurwitz. Since $\varphi(s)$ belongs to the sector $[K, I]$ with $I - K > 0$, satisfying the sector condition [6] which still holds when $\varepsilon$ approaches zero, it follows from Theorem 10.1 in [6] that $\Sigma_\varepsilon$ is globally stable, and hence the compensated system is also globally stable.

By linear fractional transformation [11], $Z_\varepsilon(s)$ can be expressed as

$$Z_\varepsilon(s) = \mathcal{F}_\varepsilon(N(s), T_\varepsilon(s))$$

where $\mathcal{F}_\varepsilon(N(s), T_\varepsilon(s))$ is the linear fractional transformation, $N(s)$ and $T_\varepsilon(s)$ have the state space realizations,

$$N(s) = \begin{bmatrix} A_0 & B_1 \\ C_1 & I \end{bmatrix} (1 - \varepsilon)I$$

$$T_\varepsilon(s) = \begin{bmatrix} A_\varepsilon & (1 - \varepsilon)B_\varepsilon C_\varepsilon \\ C_\varepsilon & 0 \end{bmatrix}$$

where $A_0 = [A - \varepsilon BN_2 C - \varepsilon BH - N_2 C F]$ $B_1 = [-B_0 0] C_1 = C_2 = (1 - \varepsilon) [-N_2 C H]$

The design of $\varphi(s)$ for a globally stable compensated system is transformed to one that finds $T(s)\mathcal{F}_\varepsilon(N(s), T_\varepsilon(s))$ is strictly positive real. After $T_\varepsilon(s)$ is constructed, $P(s)$ is obtained by

$$P(s) = \frac{1}{I - \varepsilon} [(I + (1 - \varepsilon)T_\varepsilon(s))^{-1} - I]$$

Since there exists a transfer matrix $T_\varepsilon(s)$ such that $Z_\varepsilon(s) = \mathcal{F}_\varepsilon(N(s), T_\varepsilon(s))$ is Hurwitz and if only if $(A_0, B_2)$ is stabilizable and $(C_2, A_0)$ is detectable [11], $(A_0, B_2)$ is stabilizable and $(C_2, A_0)$ is detectable if $Z_\varepsilon(s)$ is SPR. From the result given in the remarks following Theorem 4.1 in [10], $T_\varepsilon(s)$ can be chosen such that $Z_\varepsilon(s) = \mathcal{F}_\varepsilon(N(s), T_\varepsilon(s))$ is ESPR, which is also SPR, as summarized in the following Theorem.

Theorem 2 There exists a strictly proper transfer matrix $T_\varepsilon(s)$ such that $Z_\varepsilon(s)$ is SPR for an arbitrary small positive number $\varepsilon$, if and only if $(A_0, B_2)$ is stabilizable and $(C_2, A_0)$ is detectable and there exist positive definite matrices $W_1$, $W_3$ and matrices $W_2$, $W_4$ such that for an arbitrary small positive number $\varepsilon$, the following inequalities hold.

(i) $[A_0W_1 + W_1A_0 + B_2W_2 + W_2^T B_2' + W_3^T C_1^T (1 - \varepsilon)W_2' - B_1] < 0$

(ii) $[A_0W_1 + W_1A_0 + C_1 W_3' + B_2W_2 + W_2^T C_1] < 0$

(iii) The spectral radius $\rho(Y_2X_2) < 1$

where $X_2 = W_1^{-1}$ and $Y_2 = W_5^{-1}$. If $P(s)$ is chosen such that the inequalities (i) to (ii) are satisfied, the compensated system is globally stable. The matrices $A_\varepsilon$, $B_\varepsilon$ and $C_\varepsilon$ can be obtained as follows,

$$B_\varepsilon = MLL, C_\varepsilon = F_0, A_\varepsilon = (1 - \varepsilon)B_\varepsilon C_\varepsilon + A_0 + B_2F_0 + MLC_\varepsilon + \Delta_{FL}$$

where $F_0 = W_2W_1^{-1}$, $L = W_3^{-1}W_4$, $M = (I - Y_2^{-1}X_2)^{-1}$

$$\Delta_{FL} = -\frac{1}{2}[B_1 + ML][C_1 - B_1X_2 + (1 - \varepsilon)F_0] + MY_2R_2(X_2)^T - MY_2F_0 \left[B_1X_2 + \frac{1}{2}(1 - \varepsilon)C_1 - B_1X_2 + (1 - \varepsilon)F_0 \right]$$

$R_2(X_2) = (A_0 + B_2F_0)^T X_2 F_0 + \frac{1}{2}(1 - \varepsilon)C_1 - B_1X_2 + (1 - \varepsilon)F_0 \left[C_1 - B_1X_2 + (1 - \varepsilon)F_0 \right]$

Note that the LMI (ii) holds for an arbitrary small positive number $\varepsilon$, if there exists a positive definite matrix $W_3$ and a matrix $W_4$ such that

$$A_\varepsilon W_4 + A_\varepsilon W_3 C_\varepsilon + C_\varepsilon^T W_4^T W_2^T B_2' + C_\varepsilon^T W_2^T B_2' - 2I < 0$$

where

$$A_\varepsilon = \begin{bmatrix} A & 0 \\ -N_1 C & F \end{bmatrix}, B_1 = \begin{bmatrix} B' \\ F \end{bmatrix}, C_\varepsilon = [-N_2 C H]$$

Rewriting the left hand side of inequality (ii) gives,

$$\Gamma = \begin{bmatrix} A_\varepsilon W_4 + A_\varepsilon W_3 C_\varepsilon + C_\varepsilon^T W_4 W_2^T B_2' + C_\varepsilon^T W_2^T B_2' - 2I \\ \epsilon [A_\varepsilon W_4 - W_2^T A_\varepsilon - W_2^T C_\varepsilon - C_\varepsilon^T W_4] C_\varepsilon \end{bmatrix}$$

where

$$A_\varepsilon = \begin{bmatrix} -BN_2 C & BH \end{bmatrix}$$

Clearly, $\Gamma$ is negative definite when $\varepsilon$ approaches zero, and hence the LMI (ii) holds. Since (21) does not involve
\[ e \text{, it is convenient to use it in practice instead of the LMI (ii). If the system or the controller contains an integrator, Theorem 1 does not necessary hold. In this case, the compensator design procedure proposed in [3] for SISO systems can be used. The compensator } P(s) \text{ is chosen such that the equivalent system } \Sigma_e \text{, and hence the compensated system belongs to the class of compensated system that is globally stable. Let } G_e(s) \text{ belongs to a class of systems with one integrator satisfying that } Z_e(s) \text{ is SPR. From (12), } P(s) \text{ can be expressed in terms of } G_e(s) \text{ as follows,}
\]
\[ P(s) = (sG(s) - G_e(s))(1 + G_e(s))^{-1} \quad (23) \]
As \( \Sigma_e \) computed from (12) for \( P(s) \) given by (23) is \( G_e(s) \), the compensated system is therefore globally stable from the construction of \( G_e(s) \). The result is given in Corollary 1.

**Corollary 1** The compensated system shown in Fig. 2 with an integrator in either \( G \) or \( K \) is globally stable, if the saturation compensator \( P(s) \) is given by (23) with \( G_e(s) \) chosen from the class of systems with one integrator that are globally stable.

**4 Procedure for designing compensator \( P \)**

Following the result given in Theorem 2, the compensator \( P(s) \) is obtained by the following procedure.

**Step 1**
Choose a small positive number \( \epsilon = \epsilon_0 \), and solve LMI (i) and (ii) or (i) and (21) for positive definite matrices \( W_1, W_2 \) and matrices \( W_3, W_4 \).

**Step 2**
If the LMI obtained in Step 1 satisfy \( \rho(Y_1X_1) < 1 \), where \( Y_1 = W_1^{-1} \) and \( X_1 = W_3^{-1} \), then proceed to Step 3. Otherwise, choose another \( \epsilon \) and restart from step 1.

**Step 3**
For a small positive number \( \delta = \delta_0 \), solve the following LMI for a matrix \( X > 0 \),
\[
\begin{bmatrix}
A_eX + XA_e & C_e' - XB_e \\
C_e - B'eX & -2I
\end{bmatrix} < 0 
\quad (24)
\]
where \( A_e = A_{sat} - \delta B_0C_0, C_e = (1 - \delta) C_0 \). If there exists a matrix \( X > 0 \) satisfying the LMI (24), and for any positive scalar constant \( \delta \leq \delta_0 \), \( X \) satisfies the LMI (24), then \( Z_e(s) \) is SPR [10]. Obtain \( A_p, B_p, C_0 \) and hence \( P(s) \) from (20) for given \( Y_1, X_1 \) and other known matrices, and terminate the process. Otherwise choose another \( \epsilon \) and restart from step 1.

**5 Example**
Consider system (1) consisting of one integrator with
\[
A = \begin{bmatrix} -5 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.03 & 5 \\ 1 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0.08 & 0.1 \end{bmatrix}
\]
The controller (5) designed assuming no actuator saturation is,
\[
F = \begin{bmatrix} -8 & 0 \\ -3 & -0.06 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 6 & 0 \\ -0.3 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.6 \end{bmatrix}
\]
From the eigenvalues of the state matrix of the linear closed-loop system, it is asymptotically stable without actuator saturation, as illustrated in Fig.4 showing the closed-loop output from an initial state: \( x_0 = [5 \ 0 \ 0 \ 0] \). Let the control limits be: \( u_1^- = 1, u_1^+ = -1, u_2^- = 0.01 \) and \( u_2^+ = -0.1 \) i.e., \( u_1 \in [0.4399, 1] \) and \( u_2 \in [0.0417, 1] \).

The state and output of the saturated system without compensation shown in Fig. 5 are unstable. Applying the procedure described in section 4, the following \( P(s) \) is obtained from MATLAB,
\[
A_p = \begin{bmatrix} -5.0025 & 0.9179 & -0.0035 & 0.0042 \\ -0.0040 & -0.1334 & -0.0057 & 0.0069 \\ -6.0000 & 0.0000 & -8.0000 & 0.0000 \\ 0.2200 & -0.1000 & -3.0000 & -0.0600 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.0298 \\ 0.9999 \\ -0.0009 \\ -0.0002 \end{bmatrix}, \quad C_p = \begin{bmatrix} -0.0999 & -0.0021 & 1.0000 & 1.0001 \\ 0.0475 & 0.0436 & -0.0007 & 0.0008 \end{bmatrix}
\]
From (9), \( P(s) = C_p(sI - A_p)^{-1}B_p \). The state and output of the compensated system shown in Fig. 6 are stable. As the following positive definite matrix \( X \) is obtained that satisfies the LMI (24) for an arbitrary small positive scalar \( \delta \), \( Z_e(s) \) is SPR, and hence it follows from Theorem 1 that the compensated system is globally stable.

**6 Conclusion**
A dynamic saturation compensator for multivariable systems is presented. The conditions for the compensated system to be globally stable are derived, and a procedure for designing the saturation compensator based on the LMI is presented. It is shown that for systems containing up to one integrator, the system can still be stabilized globally by designing the saturation compensator from an equivalent system that is known to be globally stable. The results presented in the paper are illustrated by a simulation system consisting of an integrator.
References


