SENSITIVITY INTEGRAL RELATIONS FOR PASSIVE SYSTEMS BY OUTPUT FEEDBACK

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Abstract

Different integral functions of the sensitivity of a multi-input multi-output (MIMO) passive system with output feedback are discussed and interesting results are obtained. Particularly, specific calculations are derived for both the integral of the eigenvalues of the Hermitian part of the complementary sensitivity and the integral of the singular values of the logarithm of the sensitivity. In both cases, upper and lower bounds are derived for the integral of the min/max eigenvalues and the min/max singular values respectively. As shown in the paper, these bounds are explicitly calculated and are located at the min/max eigenvalues of the product of the high frequency gain matrix and the output feedback gain matrix. Furthermore, these bounds clearly indicate that the “bigger” the output feedback gain matrix, the lower the sensitivity that can be achieved over the whole range of frequencies; simultaneously, the cost is that the complementary sensitivity increases respectively.

1 Introduction

Sensitivity and complementary sensitivity are of great importance for the performance of a dynamic system since they are defined as the transfer function from disturbance to output and from measurement noise to output respectively. For single input-single output (SISO) open-loop stable systems with more than one pole-zero excess (relative degree greater than one), Bode in [1] stated that the integral of the logarithmic magnitude of the sensitivity function over all frequencies must equal zero. This integral relation suggests that in the presence of bandwidth constraint, the desirable sensitivity reduction in one frequency range must be traded off with the undesirable sensitivity increase at other frequencies (Freudenberg and Looze [6]). For MIMO systems with relative degree greater than one, Chen in [4] extended the results already given for SISO systems. Also in [4] the effects of unstable poles and non-minimum phase zeros are considered. However, passive systems or systems that can be passive by output feedback, as is the case for most of the physical systems, are open-loop minimum-phase with pole-zero excess (relative degree) exactly one (Byrnes et al [3]). The concept of passivity has played a prominent role in many areas of systems theory for many decades now. In modern control, the theory on passivity and strictly positive realness (SPR), has been crucially extended by Hill and Moylan [7], [3], and many others. For the case of linear time invariant (LTI) systems where passivity and SPR property are equivalent, Saberi et al [11] Huang et al [8] proposed design techniques for closed-loop feedback passivation. In this paper, specific calculations are derived for the integral of the eigenvalues of the Hermitian part of the complementary sensitivity and the integral of the singular values of the logarithm of the sensitivity function for the case of passive systems. As shown in the paper, both integrals depend on the eigenvalues of the product of the high frequency gain matrix and the output feedback gain matrix. For SISO passive systems it is proved that their values are complementary, in the sense that the value of the one is the negative of the other. For the MIMO case, lower bounds are derived for the maximum eigenvalue and the maximum singular value in those integrals. The values of those bounds are also complementary. From these bounds, it is clearly shown that increasing the magnitude of the output feedback gain matrix, a lower sensitivity can be achieved in the cost of increasing the complementary sensitivity over the whole range of frequencies. This constitutes a main difference between the systems that can be made passive by output feedback and all other systems i.e. it provides the possibility of selecting the desirable sensitivity level through the design of the output feedback gain-matrix. However, designing the control law, a trade-off is needed between the desirable sensitivity and the complementary sensitivity level.

2 Preliminaries and definitions

Consider the completely controllable and observable, square, LTI system
\[ \dot{x} = Ax + Bu, \quad y = Cx \]  
where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^n \) and \( A, B, \) and \( C \) are constant matrices of appropriate dimensions with \( \text{rank}(B) = \text{rank}(C) = m \) and input-output transfer function
\[ G(s) = C(sI - A)^{-1} B. \]  
A stable transfer function is SPR if and only if
\[ G(j\omega) + G^*(j\omega) > 0 \quad \forall \omega \in \mathbb{R} \]
\[ \lim_{\omega \to \infty} \omega^2 \left[ G(j\omega) + G^*(j\omega) \right] = 0. \]
The relation between an SPR transfer function and its state space realization \((A,B,C)\) is given by the well-known Kalman-Yakubovich-Popov Lemma [8]. Consider the LTI system of Fig. 1 with transfer function \(G(s)\) and output feedback gain \(K\). The signals \(r(s), d(s), n(s)\) and \(y(s)\) are the reference input, disturbance input, measurement noise and system output, respectively. Writing the system in the form
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + d \\
u &= -K(y - n)
\end{align*}
\]
or equivalently as
\[
\begin{align*}
\dot{x} &= (A - BKC)x - BKn \\
y &= Cx + d
\end{align*}
\]
and using \(A_\sigma = A - BKC\), the sensitivity function (the transfer function from disturbance to output) is given by
\[
S(s) = I - C(sI - A_\sigma)^{-1}BK.
\]
The complementary sensitivity function (the transfer function from measurement noise to output) has the form
\[
T(s) = C(sI - A_\sigma)^{-1}BK.
\]
From (3) and (4) we have
\[
S(s) + T(s) = I_w
\]
To proceed with our analysis the following assumptions are needed.

**Assumption 1:** The multiplicities of \(\sigma_i(S(s))\) and \(\lambda_i(T(s) + T''(s))\) are constant for all \(s \in \mathbb{C}_{+}\) where \(\mathbb{C}_{+} = \{s: \text{Re}(s) > 0\}\) denotes the open right-half plane, and \(\overline{\mathbb{C}_{+}}\) denotes its closure.

Assumption 1 is imposed to ensure that the singular values of \(S(s)\) and the eigenvalues of \(T(s) + T''(s)\) be differentiable in the closed right-half plane. A real function \(f: \mathbb{C} \rightarrow \mathbb{R}\) is said to be \(C^n(D)\)-differentiable in \(D \subset \mathbb{C}\) if its partial derivatives up to \(n\)-th order are continuous functions in \(D\), where \(\mathbb{C} = \mathbb{R} \times \mathbb{R}\) through the standard relation \(s = x + jy\). For a Hermitian matrix function \(A(s)\) analytic over a domain \(D \subset \mathbb{C}\) it is well known (see e.g., Kato [9]) that an eigenvalue of \(A(s)\) is \(C^n(D)\)-differentiable if the multiplicity of that eigenvalue is constant over \(D\).

As it is well-known a minimum phase system with relative degree one can be passive by output feedback [8]. This means that for this output feedback the closed-loop system is stable, i.e., matrix \(A_\sigma\) has no poles in \(\mathbb{C}_{+}\). Hence, from Eq. (3), (4) \(S(s)\) and \(T(s)\) are both analytic in \(\mathbb{C}_{+}\) and by using Assumption 1, the singular values \(\sigma_i(S(s))\) and the eigenvalues \(\lambda_i(T(s) + T''(s))\) are \(C^n(\mathbb{C}_{+})\)-differentiable for an SPR closed-loop transfer function.

**Assumption 2:** The output feedback gain matrix \(K\) is chosen so that
\[
CBK = (CBK)^T > 0.
\]
We recall that for passive systems the high-frequency gain matrix \(CB\) is symmetric and positive definite [8].

### 3 Main results

Proceeding with our main results, we will make use of Green’s formula for \(C^2(D)\)-differentiable functions \(f, g\) (see Lang [10]):
\[
\int_D \left( f \nabla^2 g - g \nabla^2 f \right) \, dx \, dy = \int_D \left( \frac{\partial f}{\partial n} - g \frac{\partial f}{\partial n} \right) \, d\mu
\]
where \(\partial f/\partial n\) denotes the directional derivative of \(f\) along the exterior normal of the boundary \(\partial D\) and \(\nabla^2 f\) denotes the Laplacian of \(f\).

Now, the first main result is given.

**Lemma 1:** For a minimum phase system of relative degree one with state space realization \((A,B,C)\) and output feedback gain matrix \(K\), satisfying Assumptions 1 and 2, the following integral relation for the singular values of the complementary sensitivity function holds
\[
\int_{\mathbb{C}_{+}} \frac{\lambda_i \left( T(j\omega) + T''(j\omega) \right)}{2} \, d\omega = \pi \lambda_i (CBK) + H_1
\]
where
\[
H_1 = \frac{1}{2} \int_{\mathbb{C}} x^2 \lambda_i \left[ T(x + jy) + T''(x + jy) \right] \, dx \, dy.
\]

**Proof:** We apply Green’s formula for the functions
\[
f(s) = \lambda_i \left( T(s) + T''(s) \right)
\]
and
\[
g(s) = \log \frac{x + s}{x - s}
\]
over the disc \(D_\sigma\) of Figure 2 (see [4]).

For the functions chosen the following relations hold
\[
\nabla^2 g = 0
\]
\[
g(j\omega) = 0
\]

and along \(I_\sigma\):
Using the above relations, Green’s formula takes the form

\[
\begin{align*}
\frac{\partial g}{\partial n} &= -\frac{\partial g}{\partial x} \bigg|_{x=0} = \frac{2\tau}{\tau^2 + \omega^2}.
\end{align*}
\]

(11)

the limit above takes the form

\[
\begin{align*}
\lim_{{R \to \infty}} R\lambda \left[ T\left(Re^{\omega}\right) + T^H\left(Re^{\omega}\right) \right] &= \\
= \lim_{{R \to \infty}} \lambda \left[ T(Re^{\omega}) + T^H(Re^{\omega}) \right]
\end{align*}
\]

where \( \lambda \) is the output feedback gain matrix.

For the other limit, we have

\[
\begin{align*}
\lim_{{R \to \infty}} I_1 &= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} 2\lambda \left( CBK \right) \cos^2 \theta \, d\theta \\
\end{align*}
\]

which yields

\[
\begin{align*}
\lim_{{R \to \infty}} I_1 &= \pi \lambda \left( CBK \right).
\end{align*}
\]

(18)

Now, we need to calculate the limit

\[
\begin{align*}
\lim_{{R \to \infty}} R\lambda \left[ T\left(Re^{\omega}\right) + T^H\left(Re^{\omega}\right) \right].
\end{align*}
\]

Taking into account Eq. (4)

\[
\begin{align*}
\lim_{{R \to \infty}} R\lambda \left[ T\left(Re^{\omega}\right) + T^H\left(Re^{\omega}\right) \right] = \\
= \lim_{{R \to \infty}} \lambda \left[ Ce^{\omega(1-R^{-1}e^{\omega}A_0)} + BK + e^{\omega K^TB^T(1-R^{-1}e^{\omega}A_0)}C^T \right].
\end{align*}
\]

Now, using the series expansion

\[
\begin{align*}
(I-A)^{-1} = I + A + A^2 + \ldots
\end{align*}
\]

for a matrix with eigenvalues inside the unit disc and introducing

\[
\begin{align*}
x^2 := \frac{1}{R}
\end{align*}
\]

so that

\[
\begin{align*}
\int_{-\infty}^{\infty} \frac{e^{\omega x^2}}{\sqrt{\pi x^2}} \, dx = 1
\end{align*}
\]

and the proof of Lemma 1 is completed. □

\[
\begin{align*}
\text{Theorem 1:} \text{ For a MIMO closed-loop passi ve linear system with output feedback gain matrix } K \text{ satisfying Assumptions } 1 \text{ and } 2 \text{ and state space realization } (A, B, C), \text{ the following integral relations hold}
\]

\[
\begin{align*}
\int_{-\infty}^{\infty} \frac{T(j\omega) + T^H(j\omega)}{2} \, d\omega &\geq \pi \lambda_{\max} (CBK) \quad (20) \\
\int_{-\infty}^{\infty} \frac{T(j\omega) + T^H(j\omega)}{2} \, d\omega &\leq \pi \lambda_{\min} (CBK). \quad (20a)
\end{align*}
\]
Proof: As shown by Boyd and Desoer [2], the functions \( \lambda_{\max}(T(s) + T''(s)) \) and \( \lambda_{\min}(T(s) + T''(s)) \) are subharmonic and superharmonic in \( \mathbb{C}_+ \), respectively. This implies that for any \( s \in \mathbb{C}_+ \),
\[
V^2 \lambda_{\min}(T(s) + T''(s)) \geq 0, \quad V^2 \lambda_{\max}(T(s) + T''(s)) \leq 0.
\]
Therefore \( H_i \geq 0, \quad H_n \leq 0 \) and Lemma 1 provides (20) and (20a).

For SISO systems, the eigenvalues \( \lambda_i(T(s) + T''(s)) \) will coalesce to \( (T(s) + T''(s)) = 2 \Re T(s) \). Since \( T(s) \) is analytic in \( \mathbb{C}_+ \), it follows that \( \Re(T(s)) \) is a harmonic function in \( \mathbb{C}_+ \). Thus, \( V^2 \Re(T(s)) = 0 \) in \( \mathbb{C}_+ \) and \( H_i = 0 \).

Hence, the following Corollary holds.

**Corollary 1:** For a SISO closed-loop passive system with negative output feedback gain \( k \) and state space realization \((A, B, C)\), we have the following integral relation
\[
\int_{\mathbb{R}^+} T(\omega) + T''(\omega) d\omega = \pi k CB.
\]

Next, the logarithmic integral of the sensitivity function will be derived.

**Lemma 2:** For a minimum phase system of relative degree one with state space realization \((A, B, C)\) and output feedback gain matrix \( K \) satisfying Assumptions 1 and 2, the following integral relation holds
\[
\int_{-\infty}^{\infty} \log \sigma_s(S(j\omega)) d\omega = -\pi \lambda/(CBK) + M_i
\]
where \( M_i = \int_{\mathbb{R}^+} x V^2 \log \sigma_s(S(x + jy)) dx dy \).

**Proof:** Applying Green’s formula of eq.(7) for the functions
\[
f(s) = \log \sigma_s[S(s)]
\]
and \( g(s) \) given by eq.(9) over the disc \( D \) of Fig. 1. With the use of Eq. (10)-(11), Green’s formula takes the form
\[
\int_{\mathbb{R}^+} \log \sigma_s[S(j\omega)] d\omega = -2 \int_{D} \frac{\partial g}{\partial n} d\mu + \int_{D} \left( f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) d\mu
\]
where:
\[
\int_{\mathbb{R}^+} f \frac{\partial g}{\partial n} d\mu = -\int_{-\infty}^{\infty} \frac{\partial g}{\partial \omega} \sigma_s[S(j\omega)] 2\pi \frac{r}{r^2 + \omega^2} d\omega.
\]
Multiplying by \( r \), taking the limits \( r \to \infty \) and \( R \to \infty \) and using Facts and Lemmas of the Appendix, Eq. (23) takes the form
\[
\int_{-\infty}^{\infty} \log \sigma_s[S(j\omega)] d\omega = \int_{-\infty}^{\infty} x V^2 \log \sigma_s[S(x + jy)] dx dy + \lim_{R\to \infty} I_1 - \lim_{R\to \infty} I_2
\]
where
\[
I_1 = \int_{-\infty}^{\infty} R \cos \theta \log \left[ \sigma_i \left( S(Re^\theta) \right) \right] d\theta
\]
and
\[
I_2 = \int_{-\infty}^{\infty} R^2 \cos \theta \log \left[ \sigma_i \left( S(Re^\theta) \right) \right] d\theta
\]

Now, we need to calculate the limit
\[
\lim_{R\to \infty} R \log \sigma_i \left( S(Re^\theta) \right).
\]
Define \( x := R^\theta \) and \( B(x) = S(Re^\theta) \). Equivalently, we must calculate
\[
\lim_{x \to \infty} \frac{\log \sigma_i (B(x))}{x}.
\]
From Eq. (3)
\[
B(x) = I - xe^{-\theta} C \left( I - xe^{-\theta} A_{cl} \right)^{-1} BK = I - xe^{-\theta} C \left[ I + xe^{-\theta} A_{cl} + o(x) \right] BK = I - e^{-\theta} CBK x + o(x)
\]
Now, we apply Lemma A3 to take the singular value expansion. So we have \( B(0) = I \) and \( \sigma_i(B(0)) = 1 \) with multiplicity \( m \), i.e. \( B(0) = I_n = \frac{1}{\pi \theta(0)} \cdot I_{n \times n} \cdot I_{n \times n} \)
and \( B_2 = -e^{-\theta} CBK \) and the singular value expansion is as
\[
\sigma_i(B(x)) = 1 - x \cos \theta \lambda_i(CBK) + o(x).
\]
Using Assumption 2 we have
\[
\sigma_i(B(x)) = 1 - x \cos \theta \lambda_i(CBK) + o(x)
\]
with logarithmic expansion
\[
\log \sigma_i(B(x)) = -x \cos \theta \lambda_i(CBK) + o(x).
\]
Thus, the limit is
\[
\lim_{x \to \infty} R \log \sigma_i \left( S(Re^\theta) \right) = -x \cos \theta \lambda_i(CBK).
\]
Substituting (28) into (26) and taking the limit \( R \to \infty \) we obtain
\[
\lim_{R \to \infty} I_1 = -\frac{\pi}{2} \lambda_i(CBK).
\]
For the other limit, using similar manipulations as those in the proof of Lemma 1 we obtain
\[
\lim_{R \to \infty} I_2 = \frac{\pi}{2} \lambda_i(CBK)
\]
and the proof of Lemma 2 is completed.

**Theorem 2:** For a MIMO closed-loop passive linear system with output feedback gain matrix \( K \) satisfying Assumptions 1 and 2 and state space realization \((A, B, C)\), the following integral relations hold
\[
\int_{-\infty}^{\infty} \log \sigma_s[S(j\omega)] d\omega \geq -\pi \lambda_{\max}(CBK)
\]
\[ \int_{-\infty}^{\infty} \log \sigma(S(j\omega)) \, d\omega \leq -\pi \lambda_{\text{max}}(CBK). \quad (31a) \]

Proof: As shown in [2], the function \( \log \sigma(S(s)) \) is subharmonic in \( \mathbb{C}_+ \), while the function \( \log \sigma(S(s)) \) is superharmonic in \( \mathbb{C}_+ \). This implies that for any \( s \in \mathbb{C}_+ \),
\[ V^2 \log \sigma(S(s)) \geq 0, \quad V^2 \log \sigma(S(s)) \leq 0. \]
Therefore \( M_i \geq 0 \), \( M_a \leq 0 \) and using Lemma 2 the proof is completed.

For SISO systems, the singular values \( \sigma(S(s)) \) will coalesce to \( |S(s)| \). Since \( S(s) \) is analytic in \( \mathbb{C}_+ \), it follows that \( \log S(s) \) is a harmonic function in \( \mathbb{C}_+ \). Thus, \( V^2 \log |S(s)| = 0 \) in \( \mathbb{C}_+ \) and \( M_i = 0 \). Hence, the following Corollary holds.

Corollary 2: For a SISO closed-loop passive system with negative output feedback gain \( k \) and state space realization \((A, B, C)\), we have the following integral relation
\[ \int_{-\infty}^{\infty} \log |S(j\omega)| \, d\omega = -\pi kCB. \]

Considering, now, for the MIMO case a suitable output feedback control law of the form \( K = kl_n \), \( k > 0 \) then the bound of \((31)\) becomes \(-k \lambda_{\text{max}}(CBK)\) which clearly indicates that the ‘bigger’ the output feedback gain, the lower the sensitivity that can be achieved over the whole range of frequencies.

Since, \( |\det S(s)| = \prod_{i=1}^{m} \sigma_i[S(s)] \), we have that
\[ \int_{-\infty}^{\infty} \log |S(j\omega)| \, d\omega = -\pi \sum_{i=1}^{m} \lambda_i(CBK) + \sum_{i=1}^{m} M_i. \]

However, since \( \det S(s) \) is analytic in \( \mathbb{C}_+ \) it holds that
\[ V^2 \log |\det S(x + jy)| = 0. \]

Hence, we have
\[ \sum_{i=1}^{m} M_i = \int_{-\infty}^{\infty} x \sum_{i=1}^{m} \log \sigma_i[S(x + jy)] \, dx \, dy = \int_{-\infty}^{\infty} V^2 \log |\det S(x + jy)| \, dx \, dy = 0 \]
and we proved the following Corollary.

Corollary 3: For a MIMO closed-loop passive linear system with output feedback gain matrix \( K \) satisfying Assumptions 1 and 2 and state space realization \((A, B, C)\), the following sensitivity integral relation holds
\[ \int_{-\infty}^{\infty} \log |S(j\omega)| \, d\omega = -\pi \text{Trace}(CBK). \quad (32) \]

References


Appendix

In this appendix a series of Lemmas from [4] and [5], used in the proofs of Lemmas 1 and 2, will be given without proof.

Lemma A1: The following limit converges uniformly on any compact set
\[ \lim_{r \to \infty} \log \left( \frac{r + s}{r - s} \right) = 2 \Re(s). \]

Lemma A2: The following limit converges uniformly on any compact set
\[ \lim_{r \to \infty} \frac{\partial}{\partial R} \left( \log \left( \frac{r + Re^i\theta}{r - Re^i\theta} \right) \right) = 2 \cos \theta. \]

Lemma A3: Suppose that \( G: \mathbb{R} \to \mathbb{C}^{n\times m} \) is real analytic in a neighbourhood \( D \) of the origin. Let the power series expansion of \( G(x) \) in \( D \: G(x) = \sum_{n=0}^{\infty} G_n x^n \) and the \( G_n \) SVD decomposition \( G_n = \tilde{S}(G(0)) V_n U_n^H + V_2 U_2^H \) where \( U_1, V_1 \in \mathbb{C}^{n\times n} \) with \( n \) being the multiplicity of \( \tilde{S}(G(0)) \), and \( [U_1, U_2] \) and \( [V_1, V_2] \) are unitary matrices. Suppose that \( \tilde{S}(G(0)) > 0 \). Then, for any \( x \in D \), we have
\[ \tilde{S}(G(x)) = \tilde{S}(G(0)) + \frac{\lambda}{2} \left( V_n^H G_1 U_1 + U_1^H G_1^H V_1 \right) + o(x). \]