A NEW APPROACH TO OFF-LINE CONSTRAINED ROBUST MODEL PREDICTIVE CONTROL

M.T. Cychowski*, B. Ding†, H. Tang† and T. O’Mahony*

*Department of Electronic Engineering, Cork Institute of Technology
Rossa Avenue, Cork, Ireland, E-mail: {meczychowski, tomahony}@cit.ie
† Institute of Automation, Hebei University of Technology
Tianjin, 300130, P.R. China, E-mail: dingbc@jsmail.hebut.edu.cn

Keywords: Robust regulation, polytopic description, off-line state feedback law, linear matrix inequalities.

Abstract

A new approach to address constrained robust regulation for systems with a polytopic uncertainty description is proposed. The infinite horizon state feedback-law is represented by a sequence of explicit control laws that corresponds to a sequence of asymptotically invariant ellipsoids constructed off-line one within another. Beginning with the smallest ellipsoid each subsequent one (and the corresponding controller) is determined by calculating the set of states that may be driven to an adjacent (smaller) ellipsoid in exactly one step. The approach is novel in that it incorporates the knowledge of control laws associated with smaller ellipsoids in the design of controllers for all subsequent ellipsoids. This algorithm not only considerably reduces the conservativeness but also preserves its attractive low computational complexity. A simulation example shows the effectiveness of the proposed technique.

1 Introduction

While the nominal model predictive control (MPC) problem for constrained linear systems has been systematically solved (see e.g. [1], [7] and references therein), the synthesizing of robust MPC for constrained uncertain systems has attracted much attention in recent years. This is mainly due to the fact that the standard MPC formulations lack robustness guarantees [2] and research has focused on developing robustly stable MPC algorithms. Depending on the model representation used, this problem can be solved in various ways (to mention a few: [5], [6] and [10]). As noted by Wan and Kothare [8] a candidate MPC solution must provide robust stability guarantees and, ideally, low computational burden, a large domain of attraction and a small value for the cost function index.

Polytopic descriptions are commonly used to represent uncertain systems as the description is convenient and solutions readily computed using linear matrix inequalities (LMI). In their seminal paper, Kothare et al. [5] proposed a solution based on a min-max optimisation problem where a single state feedback law is used to parameterize the infinite horizon sequence of control moves. The approach, though attractive, often suffers excessive computational burden. To overcome this difficulty, Wan and Kothare [9] determined a sequence of explicit control laws that corresponds to a series of asymptotically invariant ellipsoids constructed off-line one within another. The on-line optimization problem is then converted to simple switching control; the control law associated with the active ellipsoid is chosen from a look-up table and applied to the plant. Continuity of the feedback law over the state space is provided by a linear combination of two adjacent feedback laws.

This paper proposes an improved solution to this problem. In common with Wan and Kothare [9] the proposed solution begins by specifying a sequence of state vectors. The new algorithm then calculates the ellipsoid corresponding to the smallest (i.e. closest to the origin) state. Given knowledge of the smaller ellipsoids each subsequent one (and the corresponding controller) may then be optimized. This is achieved by calculating the ellipsoidal set of initial conditions that may be driven to an adjacent (smaller) ellipsoid in exactly one step. In this way, the proposed algorithm results in a control action that achieves a considerable reduction in conservativeness while maintaining the computational efficiency of the original approach [9].

Notation: An n-dimensional space of real valued vectors is denoted by \( \mathbb{R}^n \). For a vector \( x \in \mathbb{R}^n \) and positive-definite matrix \( W \), the weighted norm \( \| x \|_W \) is denoted by \( x^TWx \). \( x(k+i|k) \) is the value of a vector \( x \) at a future time \( k+i \) predicted at time \( k \). The symbol \( * \) induces a symmetric structure, e.g., when \( H \) and \( R \) are symmetric matrices, then

\[
\begin{bmatrix}
H + S + \ast & \ast \\
T & R
\end{bmatrix} \geq
\begin{bmatrix}
H + S + S^T & T^T \\
T & R
\end{bmatrix}.
\]

2 Problem formulation

Consider the following time varying and/or uncertain model

\[
x(k+1) = A(k)x(k) + B(k)u(k) \quad [A(k)B(k)] \in \Omega
\] (1)
with input and state constraints
\[ -u \leq u(k+i) \leq \bar{u}, \quad \forall i \geq 0, \quad (2a) \]
\[ -\Psi \leq \Psi x(k+i+1) \leq \Psi, \quad \forall i \geq 0, \quad (2b) \]
where \( u \in \mathcal{R}^n \) and \( x \in \mathcal{R}^r \) are the input and measurable state respectively; \( u := [u_1, \ldots, u_l]^T \), \( \pi := [\pi_1, \ldots, \pi_l]^T \), \( \psi := [\psi_1, \ldots, \psi_l]^T \) and \( \Psi := [\Psi_1, \ldots, \Psi_l]^T \) with \( u > 0 \), \( \pi > 0 \), \( i = 1, \ldots, m \) and \( \psi_j > 0 \), \( \Psi_j > 0 \), \( j = 1, \ldots, q \); \( \Psi \in \mathcal{R}^{qx} \). Moreover, it is assumed that \( \Omega \) is a polytopic set
\[ \Omega = \text{Co} \{A_i | B_i, A_2, \ldots, A_L | B_L \}, \quad (3a) \]
where \( \text{Co} \) denotes convex hull and \( [A_i | B_i] \) are vertices of the convex hull. In other words, if \( [A(k) | B(k)] \in \Omega \) then there exist \( L \) nonnegative coefficients \( \omega_l(k), l = 1, \ldots, L \) such that
\[ \sum_{l=1}^{L} \omega_l(k) = 1, \quad [A(k) | B(k)] = \sum_{l=1}^{L} \omega_l(k) [A_l | B_L] \quad \forall k \geq 0. \quad (3b) \]

This type of uncertainty has been studied for many years [3, 5], and is a flexible tool to describe uncertain linear time invariant (LTI) and linear time varying (LTV) systems.

### 2.1 On-line robust constrained MPC

The aim is to design a predictive controller that regulates system (1)–(2) to the steady state \( (x_{ss}, u_{ss}) = (0,0) \), and at each time \( k \) achieves the following robust cost index
\[ \min_{x(k)} \max_{[A(k) | B(k)] \in \Omega, \Gamma \geq 0} J_{ss}(k) \]
\[ = \sum_{i=0}^{\infty} \left[ \|x(k+i)\|_2^2 + \|u(k+i)\|_2^2 \right] \quad (4a) \]
The following constraints are imposed on (4a) for all \( i \geq 0 \):
\[ x(k+i+1) = A(k+i) x(k+i) + B(k+i) u(k+i), \]
\[ x(k) = x(k), \quad (4b) \]
\[ -u \leq u(k+i) \leq \bar{u}, \quad -\Psi \leq \Psi x(k+i+1) \leq \Psi, \quad (4c) \]
The weighting matrices \( Q \) and \( \mathcal{R} \) are assumed to be positive definite with \( (Q^{1/2}, A) \) detectable, and
\[ \bar{u}(k) = [u(k)k] \quad (4d) \]
is the sequence of control moves to be optimized. At time \( k \), only the first element of the optimal command sequence is actually implemented. At the next sampling instant, the optimization (4) is repeated with new measurements from the plant.

To facilitate a finite dimensional formulation Kothare et al. [5] proposed to parameterize the sequence of infinite control moves \( \bar{u}(k) \) into a single state feedback law, i.e., \( u(k+i) = F(k)x(k+i) \), \( \forall i \geq 0 \) and achieved quadratic stability using Lyapunov arguments. Following the approach in [5] the quadratic function
\[ V(i,k) = x(k+i)k^T P(k)x(k+i), \quad P(k) > 0, \quad \forall k \geq 0 \quad (5) \]
is defined and the following robust stability constraint imposed
\[ V(i+1,k) - V(i,k) \leq -\sum_{j=0}^{\infty} \left[ \|x(k+i+j)k\|_2^2 - \|u(k+i+j)k\|_2^2 \right], \quad \forall [A(k+i) | B(k+i)] \in \Omega, \quad i \geq 0. \quad (6) \]

For a stable closed-loop system, \( x(\infty | k) = 0 \) and \( V(\infty, k) = 0 \). Summing (6) from 0 to \( \infty \) yields
\[ \max_{[A(k+i) | B(k+i)] \in \Omega, \Gamma \geq 0} J_{ss}(k) \leq V(0,k) \leq \gamma \quad (7) \]
where \( \gamma > 0 \). Defining \( Q = \gamma P(k)^{-1} \) and \( F(k) = Q^{-1} \), then (7), (6) are satisfied if:
\[ \begin{bmatrix} 1 & * \\ x(k) & Q \end{bmatrix} \geq 0, \quad Q > 0, \quad (8) \]
\[ \begin{bmatrix} Q & * & * & * \\ A Q + B Y & Q & * & * \\ Q^{1/2} & 0 & \gamma I & * \\ \mathcal{R}^{1/2} & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \quad l = 1, \ldots, L. \quad (9) \]

Moreover, the requirement (4c) is met if there exist a pair of symmetric matrices \( \{Z, \Gamma\} \) satisfying the following LMIs:
\[ \begin{bmatrix} Z & Y \\ Y^T & Q \end{bmatrix} \geq 0, \quad Z_{jj} \leq z_{j,\text{inf}}^2, \quad j = 1, \ldots, m, \quad (10) \]
\[ \begin{bmatrix} Q & * \\ \Psi(A Q + B Y) & \Gamma \end{bmatrix} \geq 0, \quad \Gamma_{ss} \leq \psi_{s,\text{inf}}^2, \quad l = 1, \ldots, L; \quad s = 1, \ldots, q, \quad (11) \]
where \( z_{j,\text{inf}} = \min \{u_j, \bar{u}_j\} \), \( \psi_{s,\text{inf}} = \min \{\psi_j, \bar{\psi}_j\} \) and \( Z_{jj} \) \( (\Gamma_{ss}) \) is the \( j \)th (\( s \)th) diagonal element of \( Z \) (\( \Gamma \)). In this way,
problem (4) is transformed into the following linear objective minimization problem

$$\min_{\gamma \in \mathbb{R}^2} J_2(k)$$

subject to (8), (9), (10) and (11).

(12)

**Theorem 1.** Consider the uncertain system (1)–(2) at time instant k. The MPC algorithm, if initially feasible, robustly asymptotically stabilizes the closed-loop system.

**Proof.** see Kothare et al. [5].

The approach, though attractive, often suffers excessive computational burden. To overcome this difficulty, Wan and Kothare [9] characterized the solution of the robust constrained MPC as a sequence of state feedback laws associated with a series of nested positively invariant ellipsoids. The solution may be summarized in the following algorithm.

**Algorithm 1.** Consider an uncertain system (1) subject to input and output constraints (2a) and (2b).

**Stage 1:** Off-line, generate state points $x_1, x_2, \ldots, x_N$ where $x_{n+1}, h = N, \ldots, 2$ is nearer to the origin than $x_n$. Substitute $x(k)$ in (8) by $x_n$, $h = N, \ldots, 1$ and solve (12) to obtain $Q_h$, $\gamma_h$, the ellipsoids $e_h = \{x \in \mathbb{R}^n \mid x^T Q_h x \leq 1\}$ and the feedback laws $F_h = Y_h Q_h^{-1}$, $e_h \supseteq e_{h+1}$, $\forall h = N, \ldots, 2$.

**Stage 2:** On-line, if for each $x_n$, the following condition is satisfied,

$$Q_i^{-1} - (A_i + B_i F_{h_{i-1}})^T Q_i^{-1} (A_i + B_i F_{h_{i-1}}) > 0,$$

(13)

then at each time $k$ adopt the following control law

$$F(k) = \begin{cases} F(\alpha_h(k)), & x(k) \in e_h, \ x(k) \notin e_{h+1}, \\ F_1, & x(k) \in e_1, \end{cases}$$

(14)

where $F(\alpha_h(k)) = \alpha_h(k) F_h + (1 - \alpha_h(k)) F_{h+1}$, and for all $k$ the continuity condition

$$x(k)^T [\alpha_h(k) Q_i^{-1} + (1 - \alpha_h(k)) Q_{i+1}^{-1}] x(k) = 1,$$

must be satisfied with $0 \leq \alpha_h(k) \leq 1$ [9].

Compared with [5], the on-line computational burden is significantly reduced, but the optimization problem results in the control action that is less optimal. In the following section, an algorithm with improved optimality is proposed.

### 3 The improved off-line technique

In calculating $F_h$, Algorithm 1 does not consider $F_i$, $\forall i < h$. However, $F_i$, $\forall i < h$ are more optimal feedback laws (for $e_i$) than $F_h$. In the following, $Q_i$, $F_i$, $\gamma_i$ are chosen to be the same as in Algorithm 1 but a different technique to calculate $Q_h$, $F_h$, $\gamma_h$, $\forall h \geq 2$ is adopted. For $x_n$, $\forall h \geq 2$ the matrices $Q_h$, $F_h$ are computed such that, for all $x(k) \in e_h$ at the following times $x(k+i \mid k) \in e_{h-i}$, $e_{h-i} \subset e_h$ and inside of $e_{h-i}$ the control law $F_{h-i}$, $1 \leq i \leq h-1$ is utilized.

#### 3.1 Calculating $Q_2$, $F_2$

Define an optimization problem

$$\min_{u(k+i \mid k), i \geq 1} \max_{|i(i+1)\in \Omega} J_{2, \text{on}}(k)$$

subject to (4b), (4c) for all $i \geq 1$

and solve this problem by

$$u(k+i \mid k) = F_i x(k+i \mid k), \quad \forall i \geq 1.$$  

(15)

By analogy to Equation (7),

$$\max_{|i(i+1)\in \Omega} J_{2, \text{on}}(k) \leq x(k+i \mid k)^T P_i x(k+i \mid k) \leq \gamma_i,$$

(16)

where $P_i = \gamma_i Q_i^{-1}$. Problem (4a) now becomes a min-max optimization of (also refer to [4])

$$\mathcal{T}_2(k) := \mathcal{T}(k) = \|x(k)\|^2_2 + \|u(k)\|^2_2 + \|x(k+i \mid k)\|^2_2.$$  

(17)

It follows from (18) that

$$\mathcal{T}_2(k) = x(k)^T P_2 x(k) \leq \gamma_2,$$

(19)

and for the new variable, $P_2$, the condition

$$[A(k) + B(k) F_2] P_1 [A(k) + B(k) F_2] + Q + F_2^T F_2 \leq P_2,$$

(20)

must hold. Moreover, $u(k) = F_2 x(k)$ should satisfy the following hard constraints

$$-u \leq F_2 x(k) \leq \bar{u}, \quad -\bar{\Psi} \leq \Psi [A(k) + B(k) F_2] x(k) \leq \bar{\Psi},$$

$$\forall x(k) \in e_2$$

(21)
and the terminal constraint
\[ x(k + |k|) \in E_1, \quad \forall x(k) \in E_2. \]  
(22)

Equation (22) is equivalent to a simple matrix inequality
\[ [A(k) + B(k)F_i]Q_i^T[A(k) + B(k)F_i] - \leq \leq Q_i. \]

Defining
\[ Q_1 = y^2 F_i^{-1} \quad \text{and} \quad F_i = Y_i Q_i^{-1}, \]  
then (19), (20) and (22) can be transformed into the following LMI:
\[
\begin{bmatrix}
1 & * & * \\
x_k & Q_2 & * \\
\end{bmatrix} \geq 0, \quad Q_2 > 0,
\]
(23)

\[
\begin{bmatrix}
Q_1 * & * & * \\
A_i Q_i + B_i Y_i & \gamma_i P_i^{-1} & * \\
\end{bmatrix} \geq 0, \quad \gamma_i I \quad \text{for} \quad i = 1, \ldots, L.
\]
(24)

Constraint (21) is satisfied if [5]
\[
\begin{bmatrix}
Z_{2,j} Y_i & Q_i \\
Y_i^T & Q_i \\
\end{bmatrix} \geq 0, \quad Z_{2,j} \leq z_{j,\text{inf}}, \quad j = 1, \ldots, m,
\]
(25)

\[
\begin{bmatrix}
Q_{h} * & * & * \\
A_i Q_i + B_i Y_i & \gamma_i P_{h,1}^{-1} & * \\
\end{bmatrix} \geq 0, \quad \gamma_i I \quad \text{for} \quad i = 1, \ldots, L.
\]
(26)

Thus, \( Y_i, \gamma_i \) and \( \gamma_i \) can be obtained by solving
\[
\min_{\gamma_i, Y_i, Q_i, z_{j,\text{inf}}} \gamma_i
\]
subject to (23), (24), (25), (28) and (27).

### 3.2 Calculating \( Q_h, F_h \), \( \forall h \geq 3 \)

Define an optimization problem
\[
\min_{\gamma_h, Y_h, Q_h, z_{j,\text{inf}}} \gamma_h
\]
subject to (28) and solve this problem by
\[
\begin{align}
\min_{\gamma_h, Y_h, Q_h, z_{j,\text{inf}}} & \max_{u(k)^2_k, \gamma_h, \varepsilon} J_{\min}(k) \\
& \sum_{i=1}^{\gamma_h} \left[ \|x(k) + i|k|\|_2^2 + \|u(k + i|k|)\|_2^2 \right] \\
& \text{subject to (4b), (4c) for all } i \geq 1
\end{align}
\]
(29)

By induction,
\[
\max_{\gamma_h, Y_h, Q_h, z_{j,\text{inf}}} J_{\min}(k) \leq x(k + |k|) P_{h,1} x(k + |k|) \leq Y_{h-1}
\]
(30)

The proposed technique may be summarized in the following algorithm.

**Algorithm 2.** Consider an uncertain system (1) subject to input and output constraints (2a) and (2b).

**Stage 1:** Off-line, generate state points \( x_1, x_2, \ldots, x_N \) where \( x_{h-1}, h = N, \ldots, 2 \) is nearer to the origin than \( x_a \). Substitute
x(k) in (8) by x_i and solve (12) to obtain Q, Y, γ, the ellipsoid ε_k and the feedback laws \( F_i = Y_i Q_i^{-1} \). For \( x_h, h = N \ldots, 2 \), solve (38) to obtain \( \bar{Q}_h, \ Y_h, \ \gamma_h \) and \( \varepsilon_n, \varepsilon_n \supseteq \varepsilon_{h-1} \).

**Stage 2:** On-line, at each time k adopt the following control law

\[
F(k) = \begin{cases} 
F(\alpha_n), & x(k) \in \varepsilon_n, x(k) \notin \varepsilon_{n-1}, \\
F_1, & x(k) \in \varepsilon_1,
\end{cases}
\]  
(39)

where \( F(\alpha_n) = \alpha_n F_h + (1 - \alpha_n) F_{h-1} \), with

\[
x(k)^T \left[ \alpha_n Q_h^{-1} + (1 - \alpha_n) Q_{h-1}^{-1} \right] x(k) = 1 \quad \text{and} \quad 0 \leq \alpha_n \leq 1.
\]

**Remark 1.** In Algorithm 2, \( \alpha_n(k) \) is simplified to \( \alpha_n \) since \( x(k) \) can only stay in \( \varepsilon_n \) once.

**Remark 2.** The computation of a sequence of feedback laws in Algorithm 2 may potentially lead to feasibility problems in the optimization problem defined in Equation (38). This comes from the fact that for an arbitrary sequence of points \( x_1, x_2, \ldots, x_h \), the control law that drives the state from one ellipsoid to another in one step may not exist. This is however not a serious restriction as the feasibility of (38) may be attained for any sequence of points generated with a sufficiently small discretization step, i.e., \( \Delta x_h = x_h - x_{h-1}, \ h = N \ldots, 2 \).

**Theorem 2.** For system (1)-(2) and an initial state \( x(0) \in \varepsilon_n \), the off-line constrained robust MPC in Algorithm 2 robustly asymptotically stabilizes the closed-loop system.

**Proof.** When \( x(k) \) satisfies \( \|x(k)\|_{\varepsilon_{k-1}} \leq 1 \) and \( \|x(k)\|_{\varepsilon_n} \geq 1 \), \( h \neq 1 \), let \( Q(\alpha_n)^{-1} = \alpha_n Q_h^{-1} + (1 - \alpha_n) Q_{h-1}^{-1} \) and \( Z(\alpha_n) = \alpha_n Z_h + (1 - \alpha_n) Z_{h-1} \). By linear interpolation,

\[
\begin{bmatrix} Z(\alpha_n) & * \\ F(\alpha_n) & Q(\alpha_n)^{-1} \end{bmatrix} \geq 0
\]

which means that \( F(\alpha_n) \) satisfies the input constraint. Since \( F_{h-1} \) is a stable feedback law for all \( x(0) \in \varepsilon_{h-1} \), the matrix inequality

\[
\begin{bmatrix} Q_{h-1}^{-1} & * \\ A + B F_{h-1} & Q_{h-1} \end{bmatrix} \geq 0
\]

must hold. Moreover, (35) is equivalent to

\[
\begin{bmatrix} Q_{h-1}^{-1} & * \\ A + B F_{h-1} & Q_{h-1} \end{bmatrix} \geq 0
\]

Hence by linear interpolation,

\[
\begin{bmatrix} Q(\alpha_n)^{-1} & * \\ A + B F(\alpha_n) & Q_{h-1} \end{bmatrix} \geq 0
\]

which, since \( x(k) \in \varepsilon_n \), means that \( u(k) = F(\alpha_n) x(k) \) is guaranteed to drive \( x(k+1) \) into \( \varepsilon_{k-1} \) with the constraints satisfied. Other details are as in [8].

**4 Numerical example**

Consider the following uncertain system

\[
\begin{bmatrix} x_i(k+1) \\ x_j(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ \beta & 0.8 \end{bmatrix} \begin{bmatrix} x_i(k) \\ x_j(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),
\]

where \( \beta(k) \) is an uncertain parameter. Use \( 0.5 \leq \beta(k) \leq 2.5 \) to form a polytopic description and \( \beta(k) = 1.5 \) to calculate the state evolution. The new off-line technique has been applied with the following weighting matrices:

\[
Q = I, \quad R = 1,
\]

and with the following constraint on the control input

\[
|u(k + i)k)| \leq 0, \ \forall i \geq 0.
\]

The sequence of state feedback control laws is generated by choosing \( x_h = [1 + 0.001(h - 1) 0]^T \) and \( x_n = [2 \ 0]^T \).

Given an initially disturbed state \( x(0) = [2 \ 0]^T \), the state trajectories and the input responses for Algorithm 1 and Algorithm 2 are shown in Figure 1 and Figure 2 respectively, where the solid lines refer to Algorithm 2 and the dotted lines to Algorithm 1. Moreover, denoting

\[
\hat{J} = \sum \nu(i) \left[ \|y(i)\|^2 + \|u(i)\|^2 \right].
\]

then \( \hat{J}^* = 19.0345 \) for Algorithm 1 and \( \hat{J}^* = 17.3322 \) for Algorithm 2. The simulation results show that a less conservative control law is achieved with the new algorithm.

**5 Conclusions**

A new technique for the computation of off-line robust constrained model predictive control is proposed. Rather than optimizing each state feedback law by fixing the infinite-horizon control moves as a single state feedback law, each off-line state feedback law is optimized by considering that, if this law is applied at the current time, then at the following times less conservative feedback laws will be applied. This new algorithm consists of a varying horizon model predictive control, that is, the control horizon (say \( M \)) varies from \( M > 1 \) to \( M = 1 \), while the original Algorithm 1 can be taken as an approach with \( M = 1 \). By extending the horizon, the new algorithm gives more optimal control moves.
Acknowledgements

The research was funded by Enterprise Ireland under the PATs Research Programme 2000-2004 (Grant No. PRP/00/AMT/02A). The authors wish to gratefully acknowledge this support.

References


