STATIC $\mathcal{H}_\infty$ LOOP SHAPING CONTROL

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Abstract

The aim of this paper is to introduce a novel method of designing robust static output feedback controllers. A static output feedback version of the design procedure of Glover- McFarlane is presented and evaluated on numerical examples. Existence conditions for a static output loop shaping controller are given in terms of simple linear sufficient conditions. In this framework, the determination of a static, sub-optimal, $\mathcal{H}_\infty$ loop shaping controller reduces to an optimization problem over Linear Matrix Inequalities (LMIs). Two iterative algorithms are proposed to reduce the conservatism of the LMI conditions.

1 Introduction

The aim of this paper is to investigate the possibility of expanding $\mathcal{H}_\infty$ optimization in static output feedback design. $\mathcal{H}_\infty$ optimization procedures developed over the last two decades are now considered as standard tools for the control of multivariable systems. However, a drawback of modern linear control theory, including $\mathcal{H}_\infty$ control, is that the controller order is equal to that of the augmented plant in the synthesis problem. High order controllers may lead to implementation issues. In this paper, the design of a stabilizing $\mathcal{H}_\infty$ static output feedback gain is considered. The existence of a static output feedback controller is known to be equivalent to the existence of a positive definite matrix $R$ satisfying simultaneously a linear matrix inequality and a Riccati type matrix inequality. This problem cannot, in general, be cast as a Linear Matrix Inequality problem. The solution to this problem is still open and it has been demonstrated that its constrained counterpart is indeed NP-hard [3]. It is not known if one can develop a polynomial time algorithm for the unconstrained output feedback problem [3]. Fortunately, it is possible to derive sufficient Linear Matrix conditions from the necessary and sufficient conditions [7], [1], [6], [15]. The basic idea is to impose a certain structure on the Lyapunov matrix used in the synthesis procedure. Under this assumption, the feedback synthesis reduces to an LMI optimization problem.

This paper proposes alternate new LMI sufficient conditions for the determination of a stabilizing static-output feedback. The major advantage of using these new conditions is that they generalize easily to the $\mathcal{H}_\infty$ design procedure of McFarlane and Glover [14]. In this framework, two iterative algorithms are proposed to reduce the inherent conservatism of the LMI conditions of existence for a static output feedback controller.

The paper is organized as follows: The theoretical background to the static output feedback problem is given in Section 2. The main results are presented in Section 3 where the notions of static control and static robust control in the context of coprime factors are presented. Finally, the use of the algorithms is demonstrated on various numerical examples. Concluding remarks end the paper.

The notation is standard:

- $R^{m \times n}$: real $m \times n$ matrices.
- $I_n$: $n \times n$ identity matrix.
- $X > 0$ (resp. $\geq 0$): $X$ is symmetric and positive definite (resp. positive semi-definite).

2 Preliminaries

Let us consider a linear time invariant system described by state-space equations

$$
\begin{cases}
    \dot{x} = Ax + Bu \\
y = Cx
\end{cases}
$$

where $A \in R^{n \times n}$, $u \in R^m$ is the input control, $y \in R^p$ is the measured output. The pairs $(A, B)$ and $(A, C)$ are assumed to be, respectively, stabilizable and detectable, we assume that $C$ and $B$ are full rank.

Our aim is to compute a static output feedback law $u = Ky$ that ensures the stability of the closed-loop system $A_{cl} = A + BKC$.

Lemma 1 (Finsler’s lemma) Let $X$ be a given symmetric matrix and let $Z$ be a matrix such that

$$
\xi^T X \xi < 0
$$

where $X > 0$.

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for all nonzero vector $\xi$ such that $Z\xi = 0$. Then there exists a constant $\sigma > 0$ such that
\[ X - \sigma Z^T Z < 0. \]

Proof. See e.g. [4].

Lemma 2 The following statements are equivalent.

i) There exists a stabilizing static output feedback gain.

ii) There exists a positive-definite matrix $R > 0$ such that
\[
N_B^T (AR + RA^T) N_B < 0, \\
N_C^T (R^{-1} A + A^T R^{-1}) N_C < 0,
\]
where $N_B$ and $N_C$ denote bases of the null spaces of $B^T$ and $C$, respectively.

Proof. See e.g. [11].

3 Main results

3.1 Static Output Feedback Stabilization

Lemma 3 The following statements are equivalent.

i) There exists a stabilizing static output feedback gain.

ii) There exists a positive-definite matrix $R > 0$, a matrix $L$ of compatible dimension and a positive real number $\gamma$ such that
\[
AR + RA^T - \gamma BB^T < 0, \\
(A + LC) R + R (A + LC)^T < 0.
\]

Proof. Using Finsler’s lemma the conditions of statement ii) of lemma 2 can be rewritten as $\exists R > 0$, and positive real numbers $\sigma_1, \sigma_2$ such that
\[
AR + RA^T - \sigma_1 BB^T < 0, \\
AR + RA^T - \sigma_2 RC^T CR < 0.
\]
It is clear that if conditions (4) and (5) are satisfied for some positive real numbers $\sigma_1$ and $\sigma_2$, then they are also satisfied if one replaces $\sigma_1$ and $\sigma_2$ with $\gamma = \max(\sigma_1, \sigma_2)$. Now, let $L = -\gamma RC^T / 2$. With such a $L$, condition (3) reduces to condition (5) and lemma 3 is proven.

It is interesting to see that conditions of lemma 3 are linear in $R$. This contrasts with the well-known static output feedback conditions of lemma 2 which involve $R$ and $R^{-1}$.

For a given matrix $L$, such that $A + LC$ is stable, the linear conditions in $R$ and $\gamma$ of lemma 3 become sufficient conditions for the existence of a static output feedback gain.

3.2 Static output feedback algorithm

The feasibility of the linear conditions of lemma 3 depends on the choice of the matrix $L$. Let us consider the following lemma.

Lemma 4 Let $L = -YC^T$ where $Y \geq 0$ is the stabilizing solution to
\[ AY + YA^T - \alpha YC^T CY + BB^T = 0. \] (6)

For any $0 < \alpha < 2$ the matrix $A + LC$ is a stability matrix.

Proof. Since $(A, B)$ is stabilizable and $(C, A)$ is detectable, it is well-known that the above Riccati equation has a unique stabilizing positive semi-definite solution $Y$. First, note that $\alpha$ must be strictly positive to guarantee the existence of a semi-definite positive solution $Y$ for (6). Now, $(A + LC) Y + Y (A + LC)^T := Q = -BB^T - (2 - \alpha) Y C^T CY < 0$. From [21], $A + LC$ is stable if $Y \geq 0$, $Q < 0$ and $(Q, (A + LC)^T)$ is detectable. Hence, if the pair $(A + LC, Q)$ is stabilizable, one can conclude on the stability of $A + LC$. Because, $Q$ is a square and a full rank matrix, it is clear that the pair $(A + LC, Q)$ is stabilizable and therefore $A + LC$ is stable for any value of $\alpha$ in $]0, 2[$.

Remark: Lemma 4 guarantees the stability of $A + LC$ for $0 < \alpha < 2$. But, note that $A + LC$ can be stable even for $\alpha > 2$.

To stabilize the triple $(A, B, C)$ an iterative algorithm is proposed. The algorithm is based on the feasibility conditions (2) and (3) which, as mentioned earlier, guarantee the existence of a static output feedback controller. These conditions depend exclusively on the gain matrix $L$. The algorithm philosophy is to generate an $\alpha$-dependent set of matrices $L$ on which the LMI conditions (2) and (3) are checked. In practice, it is interesting to extend the range of $\alpha$ beyond 2, providing that $A + LC$ is stable. The algorithm is described as follows:

1. Set $i = 1$ and define the boolean variable $fail$. Set $fail = true$ and define $\Lambda = [\alpha_1, ..., \alpha_n]$ a row vector of $n$ logarithmically equally spaced points representing various values of $\alpha$. $n = 50$, $\alpha_1 = 0.01$ and $\alpha_n = 100$ suffice in practice.
2. while(fail)
3. Set $\alpha = \Lambda(i)$ in (6) and compute the corresponding $Y$.

4. Let $L = -YCT$ and solve the LMI feasibility problem (2) and (3) in the variables $R > 0$ and $\gamma > 0$. If the LMI conditions are feasible then stop (e.g. set $fail$=false), the triple $(A, B, C)$ is static output feedback stabilizable.

5. $i = i + 1$

6. endwhile

7. If the boolean variable $fail$ is false (e.g. the triple $(A, B, C)$ is static output feedback stabilizable), then solve the LMI problem in $K$: $(A + BKC)R + R(A + BKC)^T < 0$.

### 3.3 Static $\mathcal{H}_\infty$ Loop Shaping

Let $G_s$ be a strictly proper plant of order $n$ having a stabilizable and detectable state-space realization:

$$G_s := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

with $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$.

A left coprime factorization of $G_s = \bar{M}^{-1}\bar{N}$ is given by (see [21])

$$\begin{bmatrix} \bar{N}, \bar{M} \end{bmatrix} = \begin{bmatrix} A + LC & B \\ C & 0 \end{bmatrix}.$$

**Theorem 1** Let $L = -YCT$ where $Y \geq 0$ is the stabilizing solution to

$$AY + YA^T - YCTCY + BB^T = 0. \quad (7)$$

There exists a static controller $K$ such that

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + G_sK)^{-1}\bar{M}^{-1} \right\|_\infty < \gamma \quad (8)$$

if $\gamma > 1$ and if there exists a positive definite matrix $R$ solving the inequalities

$$R(A + LC)^T + (A + LC)R < 0 \quad (9)$$

$$\begin{bmatrix} AR + RA^T - \gamma BB^T & RC^T \\ CR & -\gamma I_p \\ -L^T & I_p & -\gamma I_p \end{bmatrix} < 0. \quad (10)$$

**Proof.** See [16].

The conditions of theorem 1 are sufficient for the existence of a static $\mathcal{H}_\infty$ loop shaping controller. To reduce, the conservatism of these conditions, one can suggest to iterate on the matrix $L$. The algorithm philosophy is the same as in the pure stabilization case (section 3.2), i.e. it is matter to generate an $\alpha$-dependent set of matrices $L$ on which the LMI conditions (9) and (10) will be checked.

### 3.4 LMI solution and controller reconstruction

1. Classical loop shaping: Select $W_1$ and $W_2$ to get a desired open loop shape. The nominal plant $G$ and the shaping functions $W_1$ and $W_2$ are combined to form the shaped plant $G_s = W_2GW_1$. Let $(A, B, C)$ be a realization of $G_s$. Compute the full-order McFarlane-Glover solution (for instance, by using the commands given in [2]). Tune $W_1$ and $W_2$ so that the closed-loop $H_\infty$ attenuation level $\gamma_{opt}$ is satisfactory, see e.g. [18] and [21] for a comprehensive treatment on weighting function selection. Next, these weights are used in the static output version of the design problem.

2. Set $i = 1$ and define the boolean variable $fail$. Set $fail = true$ and define $\Lambda = [\alpha_1, ..., \alpha_n]$ a row vector of $n$ logarithmically equally spaced points representing various values of $\alpha$. $n = 50, \alpha_1 = 0.01$ and $\alpha_n = 100$ suffice in practice.

3. Solve (7) and compute the matrix $R > 0$ and the scalar variable $\gamma$ solution to the LMI system (9) and (10). If these LMI conditions are feasible then set $fail$ = false (i.e. the triple $(A, B, C)$ is static output feedback stabilizable).

4. while(false)

5. Set $\alpha = \Lambda(i)$ in (6) and compute the corresponding $Y$.

6. Let $L = -YCT$ and solve the LMI feasibility problem (9) and (10) in the variables $R > 0$ and $\gamma > 0$. If the LMI conditions are feasible then stop (i.e. set $fail$=false), the triple $(A, B, C)$ is static output feedback stabilizable.

7. $i = i + 1$

8. endwhile

9. if (NOT fail)

10. The controller $K$ can be computed by making use of the Lyapunov matrix $R$ and $\gamma$ and the analytic formulae given in [10].

11. The final feedback controller $K_{ST}$ is then constructed using the static output feedback controller $K$ with the shaping functions $W_1$ and $W_2$ such that $K_{ST} = W_1KW_2$. The order of $K_{ST}$ is equal to sum of the orders of the weighting functions.

12. endif

The Glover McFarlane approach only makes sense for normalized coprime factors of the plant ($\alpha = 1$). When $\alpha \neq 1$, the algorithm returns a $\gamma$ which may not be
a good robustness/design indicator. In this case, the $\mathcal{H}_\infty$ norm of the closed-loop must be re-evaluated with the controller $K_{ST}$ and the augmented plant used in the classical Glover-McFarlane design procedure [14].

4 Examples

4.1 Static Stabilization Example

Let us consider the following simple example

$$G_s = \frac{-s + 1}{(\epsilon s + 1)(s - 2)}.$$  

In [20] it is shown that this plant is static output feedback stabilizable for $-0.5 < \epsilon \leq 0$. Solving the conditions of section 3, with $\epsilon = .45$, and $\gamma = 0.1$, we get a feasible $K = -1.995$ .

4.2 Static Glover-McFarlane $\mathcal{H}_\infty$ loop shaping controller

This example is adapted from [17]. The plant transfer function is given by

$$G(s) = \frac{200}{10s + 1} \frac{0.001s + 1}{(0.005s + 1)^2}$$

the weights were chosen as

$$W_1 = \frac{s + 2}{s}, \quad W_2 = 1.$$  

The synthesis procedure described in section 3 was implemented using the solver in the LMI toolbox [8]. In this case, after one iteration ($\alpha = 1$) the algorithm returns the following static controller

$$K = -0.502,$$

with $\gamma = 5.42$.

The controller of interest is $K_{ST} = W_1KW_2$, a 1st order controller (the static-based $\mathcal{H}_\infty$ loop shaping controller).

The full order loop shaping controller ($K_{FO}$) was designed with the same weighting functions $W_1$ and $W_2$. As expected, the best $\gamma$ attenuation level is achieved with the full order controller $K_{FO}$ for which $\gamma = 2.31$. The closed-loop sensitivity functions $(I + GK_{ST})^{-1}$ and $(I + GK_{FO})^{-1}$ for the static-based controller and the full order controller are given in figure 1. Clearly, the static and full order loop shaping controllers present similar responses.

4.3 Benchmark examples from COMPl_cib 1.0.

COMPl_cib consists of examples collected from the engineering literature and contains models from real-world application as well as pure academic models [13]. The current version of COMPl_cib contains about 80 plants which are static output feedback stabilizable. The library is useful for testing linear and non-linear semi-definite optimization solvers such as SeDuMi [19] and PENBMI [12].

The static version of the Glover and McFarlane design procedure described in this paper has been tested on the Aircraft Models (AC), Helicopter Models (HE), Jet Engine models (JE), Reactor Models (REA) and Decentralized Interconnected Systems (DIS) which are available in COMPl_cib. This represents a collection of about 40 plants. For each plant, the static version of the Glover and McFarlane design procedure has been applied with $W_1$ and $W_2$ equal to the identity matrix as we were just interested in testing the ability of our algorithm to find a static feedback controller. For any of these plants, the algorithm of section 3.3 successfully found a static output feedback controller.

5 Conclusions

In this paper, a simple method for static output feedback control synthesis is presented. The method has been extended to the well-known McFarlane and Glover $\mathcal{H}_\infty$ design method. The effectiveness of this iterative synthesis method has been demonstrated on various numerical examples. Unlike previous algorithms, the algorithm does not require $(A, B, C)$ to be a minimum phase plant [20], [1], [9]. The proposed algorithm is also much simpler than other
iterative algorithms [11], [5]. It worth mentioning that the static version of the Glover-McFarlane design procedure, presented in this paper, has been used to design a low order controller for the Bell 205 helicopter and this controller has been successfully implemented and flight tested [16].

Future work is necessary to clarify the relationship between the choice of the gain matrix $L$ and the feasibility of the LMI conditions. Future work also should focus on a possible extension of the proposed method to robust gain-scheduling control.

References


