FREQUENCY WEIGHTED $\mathcal{L}_2$ GAIN OPTIMISATION FOR IMPROVED ANTI-WINDUP PERFORMANCE

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Abstract: This work presents a method for frequency weighted $\mathcal{L}_2$ gain optimisation in the design of full-order anti-windup compensators according to the framework developed in Weston and Postlethwaite (2000). This concept provides additional scope for the tuning of compensators to extract closer tracking performance and/or the attenuation of undesirable frequency content in the output.

1. INTRODUCTION

The use of anti-windup (AW) compensation has now become common for systems which are predominantly linear but suffer from occasional plant input saturation. These linear conditioning methods become active during saturation to maintain stability in addition to ensuring a swift return to linear behaviour, hence minimising the loss of performance associated with the saturation event. In industry they are popular because their use does not restrict the nominal linear controller design process.

Traditionally, anti-windup compensators have been designed using intuition and experience and in a fairly ad hoc manner. As a result, many of these “classical” anti-windup techniques have few stability and performance guarantees associated with them. A good summary of some of these techniques can be found in Edwards and Postlethwaite (1998) and a scheme unifying many of these was developed in M.V. Kothare (1994). Since the 1990’s there have been a number of new developments in anti-windup compensation, many associated with linear matrix inequalities (LMIs) which provide a convenient way to formulate many anti-windup problems. Crucially, the use of LMIs and Lyapunov based tools allowed researchers (Marcopoli and Phillips, 1996; Nguyen and Jabbari, 1999; C.C. Cao and Ward, 2002) to develop constructive anti-windup techniques which guaranteed stability of the (nonlinear) saturated system.

More recently, research has appended performance guarantees to these anti-windup methods and there now exist a number of papers which have proposed methods that allow a systematic synthesis of such anti-windup compensators (E.F. Mulder and Morari, 2001; Saeki and Wada, 2002; Crawshaw, 2003; Turner and Postlethwaite, 2004). The problem with some of these methods is that it is sometimes possible - for simple systems at least - to find manually an AW compensator which provides improved performance when compared to the more systematic design methods. Part of the reason for this is that the stability guarantees obtained using the more formal, systematic methods are generally conservative, ultimately yielding poorer performing AW compensators. Furthermore, the inherent lack of “sharpness” of nonlinear tools when compared to their linear counterparts, makes it difficult to capture performance goals in a completely convenient way. Within the last few years, methods have begun to emerge which address the problem of improving the performance of these systematic design methods, while not forfeiting stability guarantees (M.C. Turner and Postlethwaite, 2005; G. Herrmann and Postlethwaite, 2004; Zaccarian and Teel, 2004) although such tools suffer from greater complexity and are still in their infancy.

The most commonly accepted notion of anti-windup performance is that of a swift return to linear performance i.e. recovery of the output response which would have been observed had saturation not taken place. Provided that the linear controller is well designed, return to linear performance also implies the

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While research has shown that minimisation of the $L_2$ gain is a sensible and successful approach at designing anti-windup compensators, such an approach may not capture all the desirable features of an anti-windup design. Coupled with the fact that it is sometimes possible to produce by hand a design which outperforms an optimal compensator in simulations, there is an obvious challenge to find alternative optimisations which make full use of the performance potential.

This paper uses the anti-windup framework of Weston and Postlethwaite (2000) as a basis from which to construct an alternative method for designing anti-windup compensators. In the method of Weston and Postlethwaite (2000), anti-windup is interpreted as choosing a stable linear transfer function $M(s)$ in order to stabilise the nonlinear system and improve performance.

It can be shown that a number of standard anti-windup schemes are special cases of this parameterisation, although the most effective choice of $M(s)$ seems to be that of a particular coprime factorisation of the plant. This allows both performance and stability to be addressed in a single convex optimisation and yields a compensator of the same order as of the plant (M. Turner and Postlethwaite, 2003; M.C. Turner and Postlethwaite, 2004).

The idea explored in this paper is that the $L_2$ gain can be too vague a performance measure and that it may not capture all the desirable features of an anti-windup design. Coupled with the fact that it is sometimes possible to produce by hand a design which out-performs an optimal compensator in simulations, there is an obvious challenge to find alternative optimisations which make full use of the performance potential.

2. A DECOUPLED ANTI-WINDUP SCHEME

The anti-windup compensated closed loop using the Weston and Postlethwaite method is shown in Figure 1 where the plant $G(s) = \begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix}$ and two degree of freedom controller $K(s) = \begin{bmatrix} K_1(s) & K_2(s) \end{bmatrix}$ are described by the following state space models

$$K(s) \sim \begin{bmatrix} A_K & B_K & B_{Kx} \\ C_K & D_K & D_{Kx} \end{bmatrix}$$

$$G(s) \sim \begin{bmatrix} A_G & B_G & B_{Gy} \\ C_G & D_G & D_{Gy} \end{bmatrix}$$

By choosing $\Theta_1(s) = M(s) - I$ and $\Theta_2(s) = G_2(s)M(s)$ this block diagram has been shown to be mathematically equivalent to the de-coupled representation shown in Figure 2 (Weston and Postlethwaite, 1997).

Analysis of the de-coupled anti-windup closed loop reveals that provided the linear closed loop is stable, and $G_2M$ is stable, the problem of guaranteeing asymptotic stability of the anti-windup closed loop is reduced to that of ensuring stability of the nonlinear loop. Furthermore, performance and $L_2$ stability are guaranteed by ensuring a finite $L_2$ gain exists for the mapping $T_p : u_{lin} \mapsto y_{lin}$. A particularly attractive choice for $M(s)$ appears to be that of a coprime factorisation of the plant, viz $G_2(s) = N(s)M(s)^{-1}$. In this case the disturbance filter simply becomes $N(s)$. If we consider coprime factorisations of order equal to the plant, a state space realisation for the map $T_p$ can be given as

$$\begin{bmatrix} M(s) - I \\ N(s) \end{bmatrix} \sim \begin{cases} x_p = (A_p + B_pF)x_p + B_p\tilde{u} \\ u_d = Fx_p \\ y_d = (C_p + D_pF)x_p \end{cases}$$

where $\tilde{u} = Dz(u)$. Note that the choice of $M(s)$ has now been reduced to that of choosing a state-feedback gain, $F$. In M.C. Turner and Postlethwaite (2004), the Circle Criterion was used to derive a linear matrix inequality which can be used to construct suitable anti-windup compensators. In that work the deadzone function is modelled using the sector bound

$$\tilde{u}W[u - \tilde{u}] \geq 0, \quad W = \text{diag}(w_1, \ldots, w_m) > 0$$

and then a quadratic Lyapunov function $V(x_p) = x_p^TPx_p > 0$ is sought such that

$$\dot{V}(x_p) + 2\tilde{u}W[u - \tilde{u}] < 0$$

The existence of such a function implies quadratic stability of the nonlinear loop in Figure 2. Furthermore as $M(s), N(s)$ are coprime, this ensures asymptotic...
stability of the map $T_p$. In the same paper, an $L_2$ gain term was included in order to minimise the norm of
\[ \|T_p\|_{1,2} < \gamma \]
This holds if there exists a quadratic Lyapunov function such that the following inequality holds
\[ \dot{V}(x_p) + 2\dot{u}^T W [u - \bar{u}] + \|y_d\|^2 - \gamma^2 \|u_{lin}\|^2 < 0 \]
Simple algebra shows that the above inequality is equivalent to the following matrix being negative definite
\[ \begin{bmatrix} (A_p + B_p F)P + P(A_p + B_p F)P B_p - F^T W (C_p + D_p F) \ast & -2W \ast & \ast \ast \ast & \ast \ast -\gamma I \ast \ast \ast \ast & \ast \ast \ast \ast -\gamma I \end{bmatrix} < 0 \]
where $P > 0$, $F$, diagonal $W > 0$ and scalar $\gamma > 0$ are the variables. Using standard tools from convex optimisation (Schur complement, congruence transformations) it can be shown that this is equivalent to the LMI below:
\[ Q A_p + L' B_p + A_p Q + B_p L B_p V - L' 0 \quad Q C_p + L' D_p \]
\[ \ast \quad -2V \ast \ast \ast \ast -\gamma I \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast -\gamma I \]
where the variables are $Q > 0$, $L$, diagonal $V > 0$ and scalar $\gamma > 0$. It has been shown in M.C. Turner and Postlethwaite (2004), that an anti-windup compensator ensuring $\|T_p\|_{1,2} < \gamma$ can then be constructed as described in equation (3) with $F = LQ^{-1}$. In the next section we shall show how this structure can be exploited in the design of frequency weighted compensators.

3. FREQUENCY WEIGHTED ANTI-WINDUP

In frequency weighted anti-windup we augment the nominal plant, $G$, with a frequency weight as shown in Figure 3. In this figure the physical measurement from the plant, $y$ is augmented with a frequency weighted version, $y_{W1}$, producing a stacked output vector $\tilde{y}$. $W_1$ represents the frequency weight chosen by the designer - which we use to improve performance. $G = [\hat{G}_1 \hat{G}_2]$ is now the real, “physical” plant and $G = [\hat{G}_1 \hat{G}_2]$ is the augmented plant (see Fig. 3). $W_1$ and $\hat{G}_2$ are described by the following state-space equations
\[ \dot{\hat{G}_2} = \begin{cases} \dot{x}_p = \hat{A}_p x_p + \hat{B}_p u_{lin} \\ \tilde{y} = \hat{C}_p x_p + D_p u_{lin} \end{cases} \]
\[ W_1 = \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 y \\ y_{W1} = C_1 x_1 + D_1 y \end{cases} \]
and hence
\[ G_2 \sim \begin{bmatrix} \dot{\hat{x}}_p = \hat{A}_p \hat{x}_p + \hat{B}_p u_{lin} \\ \tilde{\hat{y}} = \hat{C}_p \hat{x}_p + D_p u_{lin} \end{bmatrix} \]
\[ G \sim \begin{bmatrix} \dot{\hat{x}}_1 = \hat{A}_1 \hat{x}_1 + \hat{B}_1 y_{W1} \\ y_{W1} = \hat{C}_1 \hat{x}_1 + \hat{D}_1 y_{W1} \end{bmatrix} \]
where $x_p = [\hat{x}_p^T \hat{x}_1^T]^T$, and $\tilde{\hat{y}} = [\hat{y}_1^T \hat{y}^T]$. $M(s), N(s)$ are chosen again as co-prime factors of $G_2$ which is now the “augmented” plant. The decoupled representation of the anti-windup closed loop using the augmented plant is shown in Figure 4. Note that $y_{lin}$ and $y_d$ have now become the vector signals $\tilde{y}_{lin} = [y_{lin} y_{W1,lin}]^T$ and $\tilde{y}_d = [y_d y_{W1,d}]^T$ in which $y_{W1,lin}$ and $y_{W1,d}$ represent frequency weighted versions of the linear plant output and disturbance filter output respectively.

Whereas previously we minimised the norm of the map from $u_{lin}$ to $y_d$, we now propose to minimise the frequency weighted version of this, i.e. the map from $u_{lin}$ to $y_{W1,d}$, using the state-space realisation given in equation (6). This can be achieved by ensuring that
\[ \|W_{f,d} \tilde{y}_d\|^2 < \gamma^2 \|u_{lin}\|^2 \]
or equivalently
\[ \tilde{y}_d W_f y_{W1,d} - \gamma^2 u_{lin}'u_{lin} < 0 \]
where $W_f$ has the form
\[ W_f = \begin{bmatrix} \epsilon I & 0 \\ 0 & W_p \end{bmatrix} \]
Typically $\epsilon$ would be chosen as a small number to ensure that $y_d$ would have little impact on the design and $W_p$ is a weighting matrix used to trade-off the importance of the various channels in $y_{W1}$. Note that such a form of compensator fits exactly into the standard framework except, the plant has an extra (fictitious) output and the controller has an extra (fictitious) input as illustrated in Figure 4. In this figure, the “augmented” controller is denoted $K(s) = [K(s) \quad 0]$ and has a null input column representing the zero contribution from $y_{W1}$ in the actual system.

In a similar way to before, except using the state-space realisation of equation (5), an AW compensator which guarantees stability and $L_2$ gain of the map $T_p : u_{lin} \mapsto W_{f,d}$ can be found by solving the LMI
\[ Q A_p' + L' B_p' + A_p Q + B_p L B_p V - L' 0 \quad Q C_p + L' D_p' \]
\[ \ast \quad -2V \ast \ast \ast \ast -\gamma I \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast -\gamma W_f^{-1} \]
Note here however that because $M(s), N(s)$ are co-prime factors of the augmented plant (i.e. that including the weight $W_1(s)$), the order of the anti-windup compensator is greater than that of the real plant,
A model of an electrical network is taken from G. Grimm and Zaccarian (2003a) where Mode I represents normal linear operation, Mode II represents a period of saturation until the point at which $u$ reduces to and remains at zero, and Mode III represents final convergence to linear performance in which the compensator states relax to zero.

### 4.1 RC Circuit model

A model of an electrical network is taken from G. Grimm and Zaccarian (2003a). Figure 5 shows the circuit diagram where $V_i$ is the plant input voltage, and $V_o$ is the plant output voltage. The state space model of the plant and a robustly stabilising PID-type controller is given as:

$$\dot{K}(s) \sim \begin{bmatrix} -80 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 20.25 & 1600 & -80 & 80 \end{bmatrix}$$

$$\hat{G}_2(s) \sim \begin{bmatrix} -10.6 & -6.09 & -0.9 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -11 & -30 & 0 \end{bmatrix}$$

Notice that the matrices of the plant output equation are all negative, hence the plant output is in the opposite direction to that of the applied control signal.

For this example, the performance of the anti-windup compensator is limited by slow poles in the open loop plant, which form the basis of the poles of the disturbance filter in the compensator. This causes mode III to be significantly longer than mode II, as shown in Figure 6. Inspection of the step response of the standard anti-windup closed loop system (Figure 6) reveals that the control performance during mode II cannot be improved upon, as for the duration of mode II, the plant input is saturated in an effort to attain the reference demand. Therefore, any performance gains to be achieved must take effect during mode III. During mode III, $u_{fin}$ is a low frequency signal and hence an intelligent choice for the frequency weighting filter may be a low pass filter of an appropriately low bandwidth. Figure 6 shows a comparison between the standard compensator and that designed using the frequency weight

$$W_1(s) = \frac{0.35}{s + 0.35}$$

The application of frequency weighting gives rise to a swifter return to linear performance through its effect during mode III and the performance during mode II is unaffected.

### 4.2 Missile example

A simplified model of the roll-yaw channel dynamics of a bank to turn missile and LQG/LTR type autopilot controller is taken from Rodriguez and Cloutier (1994). As shown in Figure 7, the standard full order compensator performs very well with the result that linear performance is almost completely recovered in the second channel. The main problem with this design is the undesirable oscillation which appears during mode III when the energy stored in the compensator is dissipated through the disturbance filter.

In order to suppress these oscillations a high pass filter was used to penalise these frequencies in the $L_2$ gain optimisation. Figure 8 shows how the oscillations in the output are attenuated as a result of the frequency weighted design using the filter described by equation...
This performance improvement does come at the cost of larger compensator poles but the frequency weight can be altered to give a compromise between performance and pole magnitude.

\[
W_1(s) = \frac{s + 19}{s + 30} \quad (7)
\]

It is found that using a low pass filter frequency weight can be useful in obtaining closer reference tracking during saturation. A first order low pass filter described by equation (8) results in the performance shown in Figure 9. Note that while the tracking is slightly closer during mode II, the oscillations during mode III are of greater amplitude as they are beyond the bandwidth of the filter. In this example it is arguable that the costs outweigh the benefits but it does demonstrate the concept.

\[
W_1(s) = \frac{0.1s + 0.1}{s + 0.1} \quad (8)
\]

4.3 Remarks on tuning the compensator

The extra freedom in this method compared to other anti-windup methods is the flexibility in choosing the weight \( W_1 \), allowing explicit frequency weighting of the design. However, in contrast to standard \( H_\infty \) design, there may be some difficulty in tuning the weight \( W_1 \). There are several reasons for these difficulties:

- There is some conservatism in the method. A strong influence on the success of frequency shaping is imposed by the bounded real lemma of the open-loop plant, which can be large for some systems. As a consequence, the values of \( \gamma \) which we obtain, are therefore much greater than unity and thus our specifications are not guaranteed to be satisfied.
- In work so far, there seems to be significant “trial and error” required in the choice of the weights. In some systems, low pass filters are useful; in others, high pass filters are more appropriate. Other than observation of the system’s response, there does not seem to be a universal method of choosing the weight \( W_1(s) \). This requires further investigation.

5. CONCLUSIONS AND FUTURE WORK

This paper presents a method of tuning full-order AW compensators for improved performance by shaping an \( L_2 \) gain optimisation over frequency. This method gives the potential of improved tracking performance and the attenuation of undesirable frequency content in the output and has been validated through a number of case studies.

The order of the compensator produced using this approach is increased compared to standard designs by

\[ \text{In standard } H_\infty \text{ optimisation such as S/KS design, a } \gamma \text{ of unity indicates that our design specifications are guaranteed to have been met; anything greater than this does not provide this guarantee.} \]
the order of the filter used. Although this is undesirable, in practice, Hankel model reduction could be employed to reduce the compensator order whilst maintaining the performance benefit. For such a reduced order design, a subsequent stability check would be required as it would no longer fit the co-prime factorisation framework of the LMI. However, since stability could be considered in isolation in this case, this would allow the use of less conservative stability guarantees if necessary, for example the multivariable Popov criterion.

It would be ideal if an alternative method could be developed to weight the $L_2$ gain optimisation without introducing extra states. This would eradicate the need for model reduction and subsequent stability tests, and also facilitate the use of higher order band-pass, shelving or notch filters in the frequency weighted design with no additional cost.

Although performance improvement has been shown, this is not observed for all systems and the performance observed can be quite sensitive to small changes of the frequency weight bandwidths. For the examples presented here, the improvement is fairly modest, although this approach is expected to be more fruitful for higher order, complex systems. A practically verified example of this is the discrete anti-windup compensator of G. Herrmann and Guo (2006) which is designed for a hard disk servo system using a high pass type frequency weight. It appears that a good baseline compensator design is given using the standard full-order optimisation, but tuning for improved performance is complex. Frequency weighting is now one of a number of tools available to the designer with which to tune a successful design.

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REFERENCES


