A NOVEL DESIGN APPROACH FOR MULTIVARIABLE QUANTITATIVE FEEDBACK DESIGN WITH TRACKING ERROR SPECIFICATIONS

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Abstract: In this paper, a novel Two-Degree-Of-Freedom (2DOF) design procedure for Multi-Input Multi-Output Quantitative Feedback Theory (MIMO QFT) problems with Tracking Error Specifications (TESs) is presented. In the proposed procedure, the feedback compensator design is separated from the pre-filter design, using the model matching approach and the unstructured uncertainty modeling concept. This paper specially deals with an appropriate transformation of the MIMO system to the equivalent SISO problems, which allows easy design. Simulation results have been provided to show the effectiveness of the proposed methodology.

Keywords: Quantitative Feedback Theory (QFT), Tracking Error Specifications (TESs), Model-matching problem.

I. INTRODUCTION
Quantitative feedback theory (QFT) is a Two-Degree-Of-Freedom (2DOF) feedback control design method for uncertain plants, which may include structured and unstructured uncertainty. The general QFT problem is to design the feedback compensator and pre-filter shown in Fig. 1, to achieve the desired performance. The feedback compensator reduces the effects of uncertainties and pre-filter shifts the response to the desired region.

Fig. 1. Two-degree-of-freedom feedback system

There are two main approaches to describe the desired performance in QFT design. In most QFT problems, the desired specifications are assumed to be only on the magnitudes of the closed loop transfer functions, termed as ‘standard’ approaches in this paper. These specifications place the response of the closed loop system between two lower and upper bounds, (Borghesani, et al., 1994; Cheng, et al., 1996). In the second approach, Tracking Error Specifications (TESs) is employed (Alavi, et al., 2004; Alavi, et al., 2005; Boje, 2002; Boje, 2003; Yaniv and Chait, 1992). Constraining the magnitude of the difference between closed loop transfer function and reference model to lie within a disk around a desired response can make a better engineering sense, see (Boje, 2002). The idea of TESs for Multi-Input Multi-Output Quantitative Feedback Theory (MIMO QFT) problems was proposed by Yaniv and Chait, (1992). In order to simplify the design procedure, they selected the reference model as the pre-filter and lost one degree of freedom. Then, a 2DOF design method has been presented by Boje (2002), for multivariable systems based on relative tracking error concept. It has been shown that the zero relative tracking error assumption for the nominal plant could separate the feedback compensator design from the pre-filter design. As in standard QFT approach, the feedback
compensator is designed using a sensitivity constraint for each on-diagonal subsystem. A nonlinear search is used to find the individual elements of $F(s)$, which causes difficulties without necessarily improving the designed pre-filter. The designed pre-filter may be complex and of high order. In (Alavi, et al., 2005), a novel 2DOF design technique has been proposed for MIMO QFT problems with TESs. Using an appropriate transformation of the MIMO system to the equivalent Single-Input, Single-Output (SISO) problems, the feedback compensator design has been separated from the pre-filter design. This transformation resulted in appropriate output disturbance rejection model for the feedback compensator design. In (Alavi, et al., 2005), it has been shown how the individual elements of the pre-filter Transfer Function Matrix (TFM) can be designed for MIMO systems, using the idea suggested by Boje, (2003).

In this paper a novel 2DOF design procedure in MIMO QFT problems with TESs is presented. It is shown how the feedback compensator design is separated from the pre-filter design, using the model-matching approach and the unstructured uncertainty modeling concept. The elements of the feedback compensator TFM are designed using an appropriate output disturbance rejection model for each equivalent SISO problems. The individual elements of the pre-filter TFM are also designed via the solution of the obtained model-matching problems, i.e., the equivalent SISO problems.

The paper is organized as follows. In section II, the problem of QFT controller design is formulated. In section III, the MIMO problem is transformed to the SISO equivalent problems. In section IV and V, the pre-filter and feedback compensator design are discussed, respectively. In section VI, the design procedure is outlined and finally in section VII an illustrative example is carried out to show the effectiveness of the proposed methodology.

II. PROBLEM FORMULATION
Consider a given $N\times N$ uncertain plant with the 2DOF feedback control system shown in Fig. 1. The real TESs $E(s)$ is defined as the magnitude of difference between the reference model $M(s)$, and the TFM of the controlled system $T_m(s)$, as given by equation (1):

$$E(s)=M(s) - T_m(s)$$

$$E(s)=[e_{ij}(s)], M(s)=[m_{ij}(s)], T_m=[l_{mij}(s)] (1)$$

The objective is to design a feedback compensator, $G(s)$, and a pre-filter, $F(s)$, to meet the desired TESs, $E_d(s)$, given by equation (2) for all plants in the region of uncertainty.

$$|M(j\omega) - T(j\omega)F(j\omega)|_{ij} \leq |E_d(j\omega)|_{ij}$$

$$E_d(s)=[e_{dij}(s)]$$

$T(s)$ represents the complementary sensitivity TFM of the controlled system.

III. THE EQUIVALENT SISO PROBLEMS
In this section, the MIMO problem is transformed to the equivalent SISO problems, (Alavi, et al., 2005). It can be shown that the closed loop TFM $T_m(s)$ can be decomposed as follows:

$$T_m(s) = \alpha^{-1}(s) T(s)$$

Where,

$$\alpha(s) = \begin{bmatrix} 1 & -\alpha_{12}(s) & \cdots & -\alpha_{1N}(s) \\ -\alpha_{21}(s) & 1 & \cdots & -\alpha_{2N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{N1}(s) & -\alpha_{N2}(s) & \cdots & 1 \end{bmatrix}$$

$$\hat{T}(s) = \begin{bmatrix} l_i(s) \\ \frac{1+l_i(s)}{1+l_i(s)} \end{bmatrix} f_{ij}(s), P^{-1}(s) = \begin{bmatrix} 1 \\ q_{ij}(s) \end{bmatrix} ,$$

$$\alpha_{ij} = -\frac{q_{ij}(s)}{1+q_{ij}(s)} g_{ij}(s)$$

Substituting equation (3) into equation (1) results in:

$$e_{ij}(s) = l_i(s) q_{ij}(s) g_{ij}(s) + \hat{T}(s)$$

Using the Schwartz inequality on the right-hand side of equation (4), the part of the difference between reference model and complementary sensitivity function is separated from the other parts as follow:

$$\frac{f_{ij}(j\omega)}{i,j = 1, \ldots, N}$$

$$\begin{bmatrix} \sum_{k=1,s_{ij}}^N a_{ij}(s)L_{ij}(s) - \sum_{k=1,s_{ij}}^N a_{ij}(s)L_{ij}(s) + m_{ij}(s) - t_{ij}(s) \\ \sum_{k=1,s_{ij}}^N a_{ij}(s)L_{ij}(s) - \sum_{k=1,s_{ij}}^N a_{ij}(s)L_{ij}(s) + m_{ij}(s) - t_{ij}(s) \end{bmatrix} \leq 0$$

Defining $S_i(s) = 1/l(1+l_i(s))$ as the sensitivity function of the on-diagonal closed-loop subsystems and substituting it into $\alpha_{ij}(s)$, results in

$$\alpha_{ij}(s) = -S_i(s) q_{ij}(s) g_{ij}(s)$$

Therefore, equation (5) can be rewritten as:

$$\begin{bmatrix} f_{ij}(j\omega) \\ \sum_{k=1,s_{ij}}^N a_{ij}(s)L_{ij}(s) + m_{ij}(s) - t_{ij}(s) \end{bmatrix} \leq 0$$

Therefore, in order to achieve the desired specification (2), it is sufficient to have:
\[ S_i(j\omega) = \sum_{k=1}^{N} \left| \frac{q_{ik}(j\omega)}{q_{ik}(j\omega)} \right| \left| e_{ij}(j\omega) + m_{ij}(j\omega) \right| + m_{ij}(j\omega) \leq e_{dij}(j\omega) \text{ for } i, j = 1, \ldots, N \]

Or
\[ m_{ij}(j\omega) - t_{ij}(j\omega) \leq e_{dij}(j\omega) \text{ for } i, j = 1, \ldots, N \]

For \( i, j = 1, \ldots, N \)

As in standard QFT, the design specifications are used to over-bound any unknown functions on the right-hand side of equation (8). Using the notation \( T_{ii}(s) = l_i(s)/(1 + l_i(s)) \), \( t_{ij}(s) \) can be rewritten as \( y_{ij}(s) = T_{ii}(s)f_{ij}(s) \). Then, equation (8) can be expressed as follows:
\[ m_{ij}(j\omega) - t_{ij}(j\omega)f_{ij}(j\omega) \leq \gamma_{ij}(j\omega) \]
\( i, j = 1, \ldots, N \)

Where,
\[ \gamma_{ij}(j\omega) = E_{dij}(\omega) - |S_i(j\omega)| \sum_{k=1}^{N} \left| \frac{q_{ik}(j\omega)}{q_{ik}(j\omega)} \right| (E_{dk}(\omega) + M_{kj}(\omega)) \]

In equation (9), \( M_{ij}(\omega) \) and \( E_{dij}(\omega) \) represent the magnitude of desired design specifications, i.e., \( M_{ij}(\omega) := |m_{ij}(j\omega)| \) and \( E_{dij}(\omega) := |e_{dij}(j\omega)| \).

Equation (9) describes the MIMO problem in the form of the equivalent SISO problems. The right-hand side of (9) should be positive; therefore the following constraint on sensitivity function will be obtained as the cost paid for transformation of the MIMO problem to the equivalent SISO problem:
\[ |S_i(j\omega)| \leq \left( \sum_{k=1}^{N} \left| \frac{q_{ik}(j\omega)}{q_{ik}(j\omega)} \right| (E_{dk}(\omega) + M_{kj}(\omega)) \right)^{1/2} \]

Equation (10) also indicates a novel criterion for minimizing the interactions resulted from other subsystems. In equation (10), in order to attenuate the interactions between the subsystems, the worst case of
\[ \min_{j=1, \ldots, N} \left( \sum_{k=1}^{N} \left| \frac{q_{ik}(j\omega)}{q_{ik}(j\omega)} \right| (E_{dk}(\omega) + M_{kj}(\omega)) \right)^{1/2} \]

is considered.

IV. PRE-FILTER DESIGN

In the previous section, it has been shown that how the MIMO QFT problem (2) is transformed into the equivalent SISO problems (9) by appropriate cost on the sensitivity function for each isolated subsystem. In fact, equation (9) represents a model-matching problem in the face of uncertainty. It seems \( f_{ij}(s) = M_{ij}(s)/T_{ii}(s) \) is upstanding solution of the obtained model matching problem, but since \( T_{ii}(s) \) is uncertain and unknown, \( f_{ij}(s) = M_{ij}(s)/T_{ii}(s) \) is impractical. In addition, we will lose one degree of freedom by selecting \( f_{ij}(s) = M_{ij}(s)/T_{ii}(s) \). To solve this problem, an auxiliary function, \( J_{ii}(s) \) is added and \( f_{ij}(s) \) is suggested to design by the following equation:
\[ f_{ij}(s) = \frac{M_{ij}(s)}{T_{ii}(s)} \cdot J_{ij}(s) \]

Where, \( T_{ii}(s) \) is the nominal closed loop transfer function. The following points are to be noted in using the auxiliary function, \( J_{ij}(s) \):

1. In section V, it will be shown that by \( \gamma_{ij}(j\omega) = 1 \) for \( \omega \leq \omega_h \), the feedback compensator design can be separated from the pre-filter design which 2DOF structure is preserved. \( \omega_h \) is the performance bandwidth. In addition, this assumption on the auxiliary function at low frequency leads to easy design of \( J_{ij}(s) \) and then pre-filter.

2. To have low order pre-filters, \( J_{ij}(s) \) could be designed to cancel some poles and zeros of \( M_{ij}(s)T_{ii}(s) \). Furthermore, it is very useful for avoidance of improper realization of the designed pre-filter.

V. FEEDBACK COMPENSATOR DESIGN

Having the equivalent SISO problems and the structure of \( f_{ij}(s) \), this section presents the feedback compensator design.

By substituting the multiplicative uncertainty model of \( T_{ii}(s) \), (i.e., \( T_{ii}(s) = (1 + \Delta T_{ii}(s))T_{ii}(s) \)) and (11) into (9), we have:
\[ M_{ij}(\omega) \left[ 1 + \Delta T_{ii}(j\omega) \right] J_{ij}(j\omega) \leq \gamma_{ij}(j\omega) \]

By assuming that \( \gamma_{ij}(j\omega) \approx 1 \) for \( \omega \leq \omega_h \), equation (12) can be rewritten as (13) and the feedback compensator is separated form the pre-filter design.
\[ M_{ij}(\omega) \Delta T_{ii}(j\omega) \leq \gamma_{ij}(j\omega) \]

By relation between \( \Delta T_{ii}(s) \) and \( \Delta q_{ii}(s) \) (the multiplicative uncertainty model of \( q_{ii}(s) \)), i.e., \( q_{ii}(s) = (1 + \Delta q_{ii}(s))q_{ii}(s) \), in the following form:
\[ \Delta T_{ii}(s) = S_i(s) \cdot \Delta q_{ii}(s) \]
Equations (14) imply another constraint on the sensitivity function for each equivalent SISO problem given by:

$$\begin{align*}
[ S_i (j \omega) ] & \leq \frac{\gamma_j (j \omega)}{M_j (j \omega) [ \Delta q_{iio} (j \omega) ]} , \omega \leq \omega_h \\
& \quad \text{for } \omega \leq \omega_h , \text{ is preferred. Thus, equation (15) is modified as:} \\
\begin{align*}
[ S_i (j \omega) ] & \leq \frac{\gamma_j (j \omega)}{M_j (j \omega) [ \Delta q_{iio} (j \omega) ]_{\text{max}}} , \omega \leq \omega_h \\
\text{Substituting } \gamma_j (j \omega) \text{ into equation (16) and rearranging it, results in:}
\end{align*}
$$

$$\begin{align*}
\min_{i,j=1, \ldots, N \in \omega_h} \left\{ \frac{E_{dij} (\omega)}{M_j (j \omega) [ \Delta q_{iio} (j \omega) ]_{\text{max}} + \sum_{k=1, \ldots, N} \frac{q_{ik}}{q_{ik, \text{max}}} (E_{dij} (\omega) + M_{kj} (\omega))} \right\}
\end{align*}$$

In relation (17), in order to attenuate the interactions between the subsystems, the worst case of

$$\begin{align*}
M_j (j \omega) [ \Delta q_{iio} (j \omega) ]_{\text{max}} + \sum_{k=1, \ldots, N} \frac{q_{ik}}{q_{ik, \text{max}}} (E_{dij} (\omega) + M_{kj} (\omega))
\end{align*}$$

has been considered.

In the design procedure, two constraint equations (10) and (17) have been achieved on the sensitivity function of each equivalent SISO problem. Equation (10) is resulted from the transformation of the MIMO problem into the equivalent SISO problems. Equation (17) is resulted from the separation of pre-filter design and feedback compensator design for each equivalent SISO problem within the performance bandwidth. As in standard QFT design, the final composite permitted robust performance bounds on the loop function $l_i (s) = q_{ii} (s) g_{ii} (s)$ are constructed from the intersection of the equation (10) and equation (17) within the performance bandwidth. It is obvious that satisfying equation (17) results in satisfying equation (10). Therefore, equation (17) is the final sensitivity constraint to minimize the interactions and the effects of uncertainties for each equivalent subsystem which they are treated as an output disturbance, $D_1$, entering to the each equivalent SISO problems as shown in Fig. 2.

From above statements, to assure the achievement of the robust performance, it is sufficient to select output disturbance rejection model, $T_{D_1} (s)$ as given by equation (18).

![Fig. 2. Control Structure of i-th equivalent SISO problem.](image-url)
Remarks:
1- An appropriate selection of multiplicative uncertainty model, i.e., $\Delta q_{iio}(j\omega)_{\text{max}}$ and $\Delta T_{iio}(j\omega)_{\text{max}}$, reduces the design conservativeness.
2- The choice of a suitable nominal plant, in the sense that the lowest uncertainty model is attained, leads to lower over-design.
3- In this approach pre-filter may be high order. If $J_{ij}(s)$ is exactly designed such that $M_{ij}(s)/T_{iio}(s)$ and $J_{ij}(s)$ have the same poles or zeros, the pre-filter will be lower order.

VII. ILLUSTRATIVE EXAMPLE
Consider the linear time invariant plant:
\[
P(s) = \frac{1}{s+1} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad k_{11,22} \in [2,6], k_{12,21} \in [0.5,1.5]
\]
Suppose that the desired specifications are to achieve the steady state error less than 2% and settling time 5(sec).

**Step1:** desired specifications with the Following $M(s)$ and $E_d(s)$ are satisfied.
\[
M(s) = \begin{bmatrix} \frac{1}{(s+1)(s/5+1)} & \frac{0.01s}{s^2/0.2^2 + s/0.2 + 1} \\ \frac{0.01s}{s^2/0.2^2 + s/0.2 + 1} & \frac{1}{s+1}(s/5+1) \end{bmatrix}
\]
And,
\[
e_d(s)_{i,j=1,2} = \frac{s + 0.02}{s^2/0.55^2 + s/0.55 + 1}
\]
The nominal plant is chosen by maximum value of $k_j$ and $\omega_0 = 5(\text{rad/sec})$ is selected as the closed loop bandwidth.

**Step2:** $|\Delta q_{iio}(s)|_{\text{max}}$ is computed as $0.83(s/5+1)/(s/6+1)$ for $i=1,2$.

**Step3:** Computing the right hand side of equation (14) for each equivalent SISO loop, the output disturbance rejection models are selected as:
\[
T_{D_i} = \Delta T_{i2} = \frac{0.025(s^2/1^2 + s/1+1)}{(s^2/6.5^2 + s/6.5 + 1)}
\]
Fig. 3 shows the selected output disturbance rejection model for isolated subsystems.

If the Nichols envelope does not intersect the critical point (-180°,0dB), $T_{D_i}$ guarantees the robust performance and robust stability within $\omega \in [0.2,\omega_h]$. At higher frequencies, i.e., $\omega = [10,20,50]$, robust stability bound has been considered using the constraint on the complementary sensitivity matrix as
\[
T_{i} = \frac{l_i}{1 + l_i} \leq 1.3. \quad \text{Using MATLAB® QFT-Toolbox, the permitted design bounds become as Fig. 4. Fig. 4 shows that the following feedback compensator satisfies the performance and stability bounds.}
\]
\[
g_{ii} = \frac{72.25(s/42.18+1)(s/13.91+1)(s/0.4887+1)}{(s/500+1)(s/33.1+1)(s/3.991+1)(s/0.0647+1)}
\]

**Step4:** The multiplicative uncertainty model of $|\Delta T_{iio}|_{i=1,2}$ is obtained as:
\[
\frac{0.1s}{(s/8+1)}
\]
**Step5:** Selecting
\[
J_{ij}(s) = \frac{(s/13.91+1)}{(s/29.71+1)(s/488.9+1)}, \quad J_{ij}(s) = \frac{J_{ii}(s)}{(s^2+19.91s+247.9)}
\]
Equation (12) is satisfied for each element of the transfer function matrix. Finally, strictly proper pre-filter is achieved by:
\[
f_{ii} = \frac{0.85945(s^2+19.91s+247.9)}{(s+42.18)(s+5)(s+1)^2}
\]
Applying the designed controller to the system, figures 5 and 6 show satisfaction of the error specifications and unit step responses of the controlled system for several extreme plant cases. They show that the desired tracking error will be satisfied.

![Fig. 5. Error specifications of the control system, desired dashed](image1)

![Fig. 6. Unit step responses of the controlled system](image2)

**CONCLUSIONS**

This paper presents a simple design procedure in MIMO-QFT problems with TESs. It has been shown how the feedback compensator design is separated from pre-filter design through the concept of model matching problem and unstructured uncertainty. Appropriate transformation of MIMO system to the equivalent SISO problems is the other concept discussed in this paper. The individual elements of the pre-filter TFM are designed by employing the model matching solution. Heuristic algorithms such as genetic algorithm would be useful to find appropriate pre-filter TFM. The elements of the feedback compensator are also designed using appropriate output disturbance rejection model. An illustrative example has been provided to show the effectiveness of the design methodology.

**REFERENCES**


