SUBOPTIMAL TRIANGULAR CONTROLLER DESIGN METHODOLOGY FOR FULL MIMO STABLE SYSTEMS

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Abstract: This paper proposes a design methodology of triangular controllers for full MIMO stable plants. The procedure is based on an optimal design for a triangular model of the plant. Stability and appropriate performance can be achieved by adjusting a set of design parameters involved in a weighting function. The resulting controller provides integration and can be computed analytically.

Keywords: Multivariable control, optimal control, sparse systems.

1. INTRODUCTION

Classical multivariable (MIMO) control system design strategies (Goodwin *et al.*, 2001; Skogestad and Postlethwaite, 1996) do not take into account structural restrictions on the controller. This means that these controllers generate the control inputs based on information from every measurement of the process, i. e. these techniques deliver full MIMO controllers. However, different practical reasons (ease of tuning, maintenance, cabling, etc.) may make full MIMO controllers unsuitable for certain applications (Salgado and Conley, 2004) and hence, the need of imposing structural constraints on the controller appears naturally. The structural restrictions in which we are interested are those known as *sparse structural constraints* (Rotkowitz and Lall, 2002). This class of structural constraints refer to which process measurements are used to build each control input. Common control structures in this class are the decentralised (diagonal) and triangular structures. The first stage in designing a structure restricted controller is the controller structure selection (Salgado and Conley, 2004). This task is usually accomplished with aid of interaction measures such as the RGA and the PM (see (Salgado and Conley, 2004) and the references therein). In spite that there are many studies regarding this topic, the lack of results on restricted structure controller design methodologies is evident. The main difficulty behind this is that an optimal control approach yields a nonconvex optimisation problems (Qi *et al.*, 2004).

In the case of triangular controller structure, some interesting results are reported in (Qi *et al.*, 2004; Claveau and Chevrel, 2005), where numerical methodologies are developed to build triangular design procedures. However, those methods rely on the assumption that the plant is also

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triangular and do not make any considerations on how to use the proposed techniques in the full MIMO case.

This paper proposes a novel methodology for the design of linear discrete time triangular controllers for full MIMO stable systems. The proposed design procedure is based on computing an optimal triangular controller for a triangular model of the plant. This triangular controller provides integration and can be computed explicitly. The stability and performance on the loop built around the full MIMO plant can be achieved by an appropriate adjustment of a weighting function involved in the optimisation problem.

The text is organised as follows: Section 2 defines the main notation and mathematical preliminaries to be used on the remaining of the paper. In Section 3 the main problem is defined. Sections 4 and 5 describe the design methodology proposed in this paper. Finally, Sections 6 and 7 expose some illustrative examples and concluding remarks, respectively.

2. NOTATION AND PRELIMINARIES

Most of the notation used throughout this paper is standard. The $(i,j)^{th}$ element and the i^{th} column of a given matrix $\boldsymbol{A}$ are denoted as A_{ij} (or $[\boldsymbol{A}]_{ij}$ for clarity) and $\boldsymbol{A_{*i}}$ respectively. The $n \times n$ null matrix is denoted as $\boldsymbol{0_n}$ and the i^{th} element of the canonical base of $\mathbb{R}^n$ is denoted as $\boldsymbol{e_i^n}$. The complex function spaces $(\mathcal{R})\mathcal{L}_2$, $(\mathcal{R})\mathcal{H}_2$, $(\mathcal{R})\mathcal{H}_2^{\perp})$ and $(\mathcal{R})\mathcal{H}_\infty$ are defined as usual (Silva and Salgado, 2005). The norm in $\mathcal{L}_2$ is called 2-norm and denoted as $||\cdot||_2$. Given $\boldsymbol{A}(z) \in \mathbb{C}^{n\times n}$, we define its ℓ^{th} submatrix as $\boldsymbol{A_\ell}(z) \in \mathbb{C}^{(n-\ell+1)\times(n-\ell+1)}$ such that

$$[\boldsymbol{A_\ell}(z)]_{ij} = A_{(i+\ell-1)(j+\ell-1)}(z), \quad \begin{array}{l}\forall i \leq n-\ell+1 \\ \forall j \leq n-\ell+1\end{array} \tag{1}$$

Note that according to this definition, $\boldsymbol{A_1}(z) = \boldsymbol{A}(z)$ and $\boldsymbol{A_n}(z) = A_{nn}(z)$. Given a stable transfer matrix $\boldsymbol{A}(z)$, we define its generalised left unitary interactor (GLUI), $\boldsymbol{\xi_A}(z)$, as a minimum phase matrix such that the product $\boldsymbol{\xi_A}(z)\boldsymbol{A}(z)$ is biproper, minimum phase and stable. Note that since $\boldsymbol{\xi_A}(z)$ is unitary, then $||\boldsymbol{\xi_A}(z)\boldsymbol{A}(z)||_2^2 = ||\boldsymbol{A}(z)||_2^2$. An algorithm to build such interactor matrices can be found in (Silva and Salgado, 2005). In this paper we assume that all systems are linear, time invariant and discrete time.

3. PROBLEM DEFINITION

Optimal controllers must be thought in terms of the minimisation of a cost functional that is meaningful for the control objectives. A common choice (Goodwin *et al.*, 2001; Skogestad and Postlethwaite, 1996) is to consider the weighted loop sensitivity function, i. e. minimise

$$J(\boldsymbol{G},\boldsymbol{W}) = \left|\left|\boldsymbol{S_o}(z)\frac{\boldsymbol{W}(z)}{z-1}\right|\right|_2^2, \tag{2}$$

where $\boldsymbol{S_o}(z)$ is the loop sensitivity function and the matrix $\boldsymbol{W}(z)$ is biproper, stable, minimum phase and such that

$$\boldsymbol{W}(z) = \text{diag}\{W_1(z), W_2(z), \ldots, W_n(z)\}. \tag{3}$$

The weighting matrix $\boldsymbol{W}(z)$ is usually chosen to define the bandwidth of the different channels of the control loop. This can be done with a diagonal $\boldsymbol{W}(z)$, so that (3) does not pose any serious limitation to the generality of the control problems encompassed by minimisation of (2). Moreover, the weighting factor $(z-1)^{-1}$ in (2) makes the functional $J(\boldsymbol{G},\boldsymbol{W})$ well defined if and only if $\boldsymbol{S_o}(1) = \boldsymbol{0}$, which is equivalent to consider only controllers that provide integration and hence, allow perfect steady state tracking of constant reference signals.

If the plant $\boldsymbol{G}(z)$ is stable, the Youla parametrisation of all stabilising and proper controllers (Goodwin *et al.*, 2001), leads to the loop sensitivity function

$$\boldsymbol{S_o}(z) = \boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{Q}(z), \tag{4}$$

where $\boldsymbol{Q}(z) \in \mathcal{RH}_\infty$. A necessary and sufficient condition for the controller to have integral action is to force $\boldsymbol{Q}(1) = \boldsymbol{G}^{-1}(1)$. Thus an additional assumption is necessary.

Assumption 1. The plant has non singular DC gain, i. e. $\det \boldsymbol{G}(1) \neq 0$.

Additionally, we are interested in triangular controllers belonging to the class $\mathcal{S}_t$ defined as follows

$$\mathcal{S}_t = \left\{\boldsymbol{X}(z) \in \mathbb{C}^{n\times n} : X_{ij}(z) \equiv 0 \, \forall j > i\right\}. \tag{5}$$

Without loss of generality, we assume that the transfer matrices in $\mathcal{S}_t$ are lower triangular.

If the controller $\boldsymbol{C}(z)$ is constrained to be lower triangular. This is equivalent to ensure

$$\boldsymbol{Q}(z)\left(\boldsymbol{I}-\boldsymbol{G}(z)\boldsymbol{Q}(z)\right)^{-1}\in\mathcal{S}_t. \tag{6}$$

Using (2),(4) and (6), the optimal control problem of interest can be stated as that of finding $\boldsymbol{Q_{t\,opt}}(z)$ such that

$$\boldsymbol{Q_{t\,opt}}(z)=\arg\min_{\boldsymbol{Q}\in\mathcal{RH}_\infty\cap\Omega}\left\|\frac{\boldsymbol{I}-\boldsymbol{G}(z)\boldsymbol{Q}(z)}{z-1}\boldsymbol{W}(z)\right\|_2^2, \tag{7}$$

where Ω is the set of all transfer matrices $\boldsymbol{Q}(z)$ such that (6) holds. The set Ω is nonconvex (Rotkowitz and Lall, 2002; Qi *et al.*, 2004; Rotkowitz, 2005). This poses a serious difficulty to the computation of an explicit solution of the problem, so that the use of numerical search algorithms seems inescapable in the general formulation given by (7) (Qi *et al.*, 2004).

Previous research work (Silva *et al.*, 2005) show that for the stable case, if the plant is triangular itself and non singular a. e., then problem (7) is convex and can be restated as

$$\boldsymbol{Q_{t\,opt}}(z)=\arg\min_{\boldsymbol{Q}\in\mathcal{RH}_\infty\cap\mathcal{S}_t}\left\|\frac{\boldsymbol{I}-\boldsymbol{G}(z)\boldsymbol{Q}(z)}{z-1}\boldsymbol{W}(z)\right\|_2^2 \tag{8}$$

This formulation is restricted only to those plants belonging to $\mathcal{S}_t$, but its convexity allows to obtain an explicit solution to the optimisation problem.

4. OPTIMAL DESIGN FOR TRIANGULAR MODEL

Next lemma gives a preliminary result which is necessary for the derivation of the main theorem of this section.

Lemma 2. Let $\boldsymbol{G}(z)\in\mathcal{RH}_2$ be a transfer matrix without zeros on the region $|z|=1$ and let $W(z)$ be a scalar, biproper, stable and minimum phase transfer function. Consider the cost functional $J(\boldsymbol{G},W)$ defined in (2), then

$$\begin{aligned}\boldsymbol{Q_{opt}}(z)&=\arg\min_{\boldsymbol{Q}(z)\in\mathcal{RH}_\infty}J(\boldsymbol{G},W) &(9)\\ &=\left(\tilde{\boldsymbol{G}}(z)W(z)\right)^{-1}\left\{\left[\boldsymbol{\xi_G}(z)W(z)\right]_\perp\big|_{z=1}\right.\\ &\quad\left.+\left[\boldsymbol{\xi_G}(z)W(z)\right]_2\right\}, &(10)\end{aligned}$$

where $\tilde{\boldsymbol{G}}(z)=\boldsymbol{\xi_G}(z)\boldsymbol{G}(z)$, $\boldsymbol{\xi_G}(z)$ is the GLUI of $\boldsymbol{G}(z)$ and $[\cdot]_2$ ($[\cdot]_\perp$) denotes the portion in $\mathcal{RH}_2$ ($\mathcal{RH}_2^\perp$) of the argument.

Proof: According to (2) and introducing the GLUI of $\boldsymbol{G}(z)$, the cost functional can be written as

$$J(\boldsymbol{G},W)=\left\|\frac{\boldsymbol{\xi_G}(z)W(z)-\tilde{\boldsymbol{G}}(z)W(z)\boldsymbol{Q}(z)}{z-1}\right\|_2^2, \tag{11}$$

where it has been used the fact that $W(z)$ is scalar and $\tilde{\boldsymbol{G}}(z)=\boldsymbol{\xi_G}(z)\boldsymbol{G}(z)$. An orthogonal decomposition (Goodwin *et al.*, 2001) of the product $\boldsymbol{\xi_G}(z)W(z)$ can be made as

$$\boldsymbol{\xi_G}(z)W(z)=\left[\boldsymbol{\xi_G}(z)W(z)\right]_\perp+\left[\boldsymbol{\xi_G}(z)W(z)\right]_2. \tag{12}$$

Substituting (12) in (11) and using the orthogonal separation of the 2-norm, it follows that

$$J(\boldsymbol{G},W)=||\boldsymbol{A}(z)||_2^2+||\boldsymbol{B}(z)||_2^2, \tag{13}$$

where the matrix $\boldsymbol{A}(z)$ is independent of $\boldsymbol{Q}(z)$ and

$$\boldsymbol{B}(z)=\left\|\frac{\left[\boldsymbol{\xi_G}(z)W(z)\right]_\perp\big|_{z=1}+\left[\boldsymbol{\xi_G}(z)W(z)\right]_2}{z-1}-\frac{\tilde{\boldsymbol{G}}(z)W(z)\boldsymbol{Q}(z)}{z-1}\right\|_2^2. \tag{14}$$

Then, the optimal Youla parameter that minimises (13) is such that sets $\boldsymbol{B}(z)$ to zero, i. e.

$$\begin{aligned}\boldsymbol{Q_{opt}}(z)=&\left(\tilde{\boldsymbol{G}}(z)W(z)\right)^{-1}\left\{\left[\boldsymbol{\xi_G}(z)W(z)\right]_\perp\big|_{z=1}\right.\\ &\left.+\left[\boldsymbol{\xi_G}(z)W(z)\right]_2\right\}\in\mathcal{RH}_\infty,\end{aligned} \tag{15}$$

and the result follows. □

Lemma 2 gives an explicit solution to the problem of minimising the cost functional in (2) with a scalar weighting function and without structural restrictions on the controller.

Suppose that a triangular model $\boldsymbol{G_t}(z)\in\mathcal{S}_t$ of the full MIMO plant $\boldsymbol{G}(z)$ is available. Then Lemma 2 allows to solve the optimisation problem (8) for $\boldsymbol{G_t}(z)$ and hence, to obtain the optimal triangular controller for the triangular model. The following theorem is the key idea leading to the main result of this paper.

Theorem 3. Let $\boldsymbol{G_t}(z)\in\mathcal{RH}_2\cap\mathcal{S}_t$ be a transfer matrix without zeros in the region $|z|=1$ and $\boldsymbol{W}(z)=\mathrm{diag}\{W_1(z),W_2(z),\ \ldots,W_n(z)\}$ a biproper, stable and minimum phase transfer matrix such that $W_i(z)$ is scalar $\forall i=1,2,\ldots,n$. Define $\boldsymbol{Q_{i\,opt}}(z)$ as the unrestricted optimal solution of the optimisation problem in Lemma 2 using the functional $J(\boldsymbol{G_i},W_i)$, where $\boldsymbol{G_i}(z)$ is the i^{th}-submatrix of $\boldsymbol{G_t}(z)$, i. e.

$$\boldsymbol{Q}_{\boldsymbol{i}\,\boldsymbol{opt}}(z) = \arg \min_{\boldsymbol{Q}(z)\in\mathcal{RH}_\infty} J\left(\boldsymbol{G_i}, W_i\right). \quad (16)$$

Then

$$\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z) = \arg \min_{\boldsymbol{Q}(z)\in\mathcal{RH}_\infty\cap\mathcal{S}_t} J\left(\boldsymbol{G_t}, \boldsymbol{W}\right) \quad (17)$$

$$= \sum_{i=1}^{n} \text{diag}\left\{\boldsymbol{0_{i-1}}, \boldsymbol{Q}_{\boldsymbol{i}\,\boldsymbol{opt}}(z)\right\} \boldsymbol{1_{ii}}, \quad (18)$$

where the special matrix $\boldsymbol{1_{ii}} \in \mathbb{C}^{n\times n}$ has zeros in all of its elements, except in the $(i,i)^{th}$, in which has a 1.

Proof: Using elementary properties of the 2-norm and taking advantage of the facts that $\boldsymbol{W}(z)$ is diagonal and $\boldsymbol{G}(z) \in \mathcal{S}_t$, the cost functional $J\left(\boldsymbol{G}, \boldsymbol{W}\right)$ can be vectorised (Goodwin *et al.*, 2001) and expressed as

$$J\left(\boldsymbol{G}, \boldsymbol{W}\right) = \left|\left|\frac{\boldsymbol{\Lambda} - \boldsymbol{G_A}(z)\boldsymbol{Q_{vec}}(z)}{z-1}\right|\right|_2^2, \quad (19)$$

where

$$\boldsymbol{\Lambda}(z) = \Big[\boldsymbol{\Lambda_1}(z)\ \boldsymbol{\Lambda_2}(z)\ \cdots\ \boldsymbol{\Lambda_n}(z)\Big]^T, \quad (20)$$

$$\boldsymbol{Q_{vec}}(z) = \Big[\ \boldsymbol{Q_{*1}}^T(z)\ Q_{22}(z)\ Q_{32}(z)\ \cdots$$
$$Q_{n2}(z)\ \cdots\ Q_{nn}(z)\Big]^T, \quad (21)$$

$$\boldsymbol{G_A}(z) = \text{diag}\left\{\boldsymbol{G_1}(z)W_1(z), \boldsymbol{G_2}(z)W_2(z)\right.$$
$$\left., \ldots, \boldsymbol{G_n}(z)W_n(z)\right\}, \quad (22)$$

and $\boldsymbol{\Lambda_i}(z) = W_i(z)\boldsymbol{e_1^{n-i+1}}$. Expression (19) allows to separate the optimisation problem (17) in n independent problems of lower dimensions. In fact, using properties of the 2-norm, (19) can be rewritten as

$$J\left(\boldsymbol{G}, \boldsymbol{W}\right) = \sum_{i=1}^{n} \left|\left|\frac{\boldsymbol{\Lambda_i}(z) - \boldsymbol{G_i}(z)W_i(z)\boldsymbol{Q^i_{vec}}(z)}{z-1}\right|\right|_2^2 \quad (23)$$

where $\boldsymbol{Q^i_{vec}}(z) \in \mathbb{C}^{n-i+1}$ is a column vector containing every nonzero element of the i^{th} column of $\boldsymbol{Q}(z)$, i. e.

$$\boldsymbol{Q^i_{vec}}(z) = \Big[Q_{ii}(z)\ Q_{(i+1),i}(z)\ \cdots Q_{ni}(z)\Big]. \quad (24)$$

Since $\boldsymbol{\Lambda_i}(z) \in \mathcal{RH}_2$, $\forall i = 1, 2, \ldots, n$, a procedure similar to the one in Lemma 2 can be done in each term of (23). As a matter of fact, by introducing the GLUI $\boldsymbol{\xi_{G_i}}(z)$ of each submatrix $\boldsymbol{G_i}(z)$ and making the appropriate orthogonal decompositions on each term it follows

$$J\left(\boldsymbol{G}, \boldsymbol{W}\right) = \sum_{i=1}^{n} \left\{||\boldsymbol{M_i}(z)||_2^2 + ||\boldsymbol{N_i}(z)||_2^2\right\}, \quad (25)$$

where $\boldsymbol{M_i}(z)$ is independent of $\boldsymbol{Q^i_{vec}}(z)$ and

$$\boldsymbol{N_i}(z) = \frac{\left[\boldsymbol{\xi_{G_i}}(z)\boldsymbol{\Lambda_i}(z)\right]_\perp|_{z=1} + \left[\boldsymbol{\xi_{G_i}}(z)\boldsymbol{\Lambda_i}(z)\right]_2}{z-1}$$
$$- \frac{\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z)W_i(z)\boldsymbol{Q^i_{vec}}(z)}{z-1}, \quad (26)$$

with $\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z) = \boldsymbol{\xi_{G_i}}(z)\boldsymbol{G_i}(z)$. Since the product $\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z)W_i(z)$ is biproper and minimum phase, then the matrix $\left(\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z)W_i(z)\right)^{-1} \in \mathcal{RH}_\infty$ and the optimum $\boldsymbol{Q^i_{vec\,opt}}(z)$ is such that sets $\boldsymbol{N_i}(z)$ to zero, i. e.

$$\boldsymbol{Q^i_{vec\,opt}}(z) = \left(\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z)W_i(z)\right)^{-1} \left\{\left[\boldsymbol{\xi_{G_i}}(z)\boldsymbol{\Lambda_i}(z)\right]_2\right.$$
$$\left. + \left[\boldsymbol{\xi_{G_i}}(z)\boldsymbol{\Lambda_i}(z)\right]_\perp|_{z=1}\right\} \in \mathcal{RH}_\infty. \quad (27)$$

Substituting the definition of $\boldsymbol{\Lambda_i}(z)$ in the last expression, the result of Lemma 2 can be recognised as

$$\boldsymbol{Q^i_{vec\,opt}}(z) = \left(\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z)W_i(z)\right)^{-1} \left\{\left[\boldsymbol{\xi_{G_i}}(z)W_i(z)\right]_2\right.$$
$$\left. + \left[\boldsymbol{\xi_{G_i}}(z)W_i(z)\right]_\perp|_{z=1}\right\} \boldsymbol{e_1^{n-i+1}} \quad (28)$$

$$= \boldsymbol{Q}_{\boldsymbol{i}\,\boldsymbol{opt}}(z)\boldsymbol{e_1^{n-i+1}}, \quad (29)$$

where

$$\boldsymbol{Q}_{\boldsymbol{i}\,\boldsymbol{opt}}(z) = \arg \min_{\boldsymbol{Q}(z)\in\mathcal{RH}_\infty} J\left(\boldsymbol{G_i}, W_i\right). \quad (30)$$

Finally, the optimal triangular Youla parameter, $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$, can be reconstructed with an appropriate arrangement of the vectors $\boldsymbol{Q^i_{vec\,opt}}(z)$ as

$$\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}} = \sum_{i=1}^{n} \text{diag}\left\{\boldsymbol{0_{i-1}}, \boldsymbol{Q}_{\boldsymbol{i}\,\boldsymbol{opt}}(z)\right\} \boldsymbol{1_{ii}} \in \mathcal{RH}_\infty, \quad (31)$$

and the result follows. □

The form of (18) puts on evidence that the optimal triangular controller is built upon the full MIMO optimal controllers for every submatrix $\boldsymbol{G_i}(z)$ of $\boldsymbol{G_t}(z)$ considering the scalar weighting function $W_i(z)$ on each cost functional. Moreover, (18) shows that

$$\left[\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)\right]_{nn} = \arg \min_{\boldsymbol{Q}(z)\in\mathcal{RH}_\infty} J(G_{nn}, W_n), \quad (32)$$

which means that the $(n,n)^{th}$ element of the optimal triangular Youla parameter must be built based on an optimal SISO loop around $G_{nn}(z)$. This makes sense by noting that in a triangular system, the n^{th} input is the only input that affects a single output.

5. TRIANGULAR CONTROL OF FULL MIMO SYSTEMS

This section deals with the problem of using the triangular controller defined by $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$ in Theorem 3 to build a closed loop around the full MIMO plant $\boldsymbol{G}(z)$. Since the optimisation procedure is done considering only the triangular portion of $\boldsymbol{G}(z)$, neither performance nor stability are guaranteed when using $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$ to control $\boldsymbol{G}(z)$. However, it should be noticed that $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(1) = \boldsymbol{G_t}^{-1}(1)$, which implies that the triangular controller of the previous section guarantees integration in the loop around $\boldsymbol{G(z)}$ (the proof is omitted due to lack of space).

Assume that the procedure given in the previous section delivers a Youla parameter $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$ that does not stabilise the plant $\boldsymbol{G}(z)$. It is known that since $\boldsymbol{G}(z)$ is stable, this problem can be solved by reducing the bandwidth of $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$ (Goodwin *et al.*, 2001). From (18) we have that every zero of $\boldsymbol{W}(z)$ is also pole of $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$. This means that the bandwidth of $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$ can be arbitrarily diminished by choosing the zeros of $\boldsymbol{W}(z)$ sufficiently close to $z = 1$. This analysis implies that a convenient generic form for each element of $\boldsymbol{W}(z)$ is

$$W_i(z) = \frac{z - a_i}{z(1 - a_i)}, \qquad (33)$$

where $a_i \in [0, 1\,[$ and $i = 1, 2, \ldots, n$. The stability of the closed loop can then be attained by an appropriate election of the set of design parameters $\{a_i\}$, i. e. $\{a_i\}$ must be chosen sufficiently close to $z = 1$ such that, at least, stability is achieved. When this is accomplished, a fine tuning of the design parameters may be done in order to obtain certain loop performance. This procedure requires that the chosen interaction measure suggests that a triangular plant model is a sensible choice.

6. ILLUSTRATIVE EXAMPLES

Example 4. Consider a plant modeled by the transfer matrix

$$\boldsymbol{G}(z) = \begin{bmatrix} \frac{z-1.5}{z^2(z-0.9)} & \frac{-0.25(z-5)}{z^2} \\ \frac{z-1.7}{z(z-0.9)} & \frac{1}{z-0.9} \end{bmatrix}. \qquad (34)$$

The Participation Matrix (PM) (Salgado and Conley, 2004) associated to $\boldsymbol{G}(z)$ is

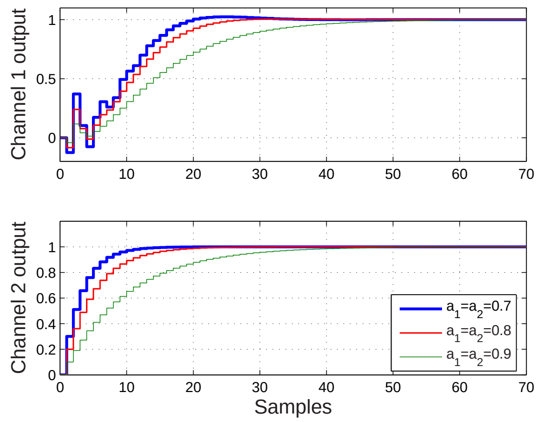

Fig. 1. Step responses for different triangular designs in Example 4.

$$\boldsymbol{\Phi} = \begin{bmatrix} 0.23 & 0.05 \\ 0.32 & 0.4 \end{bmatrix}. \qquad (35)$$

The small value of Φ_{12} suggests that a the use of lower triangular plant model for controller design is a sensible choice. Hence, we consider $\boldsymbol{G_t}(z)$ as the lower triangular portion of $\boldsymbol{G}(z)$. A first attempt to the triangular design may consider $a_1 = a_2 = 0$ which is equivalent to choose $\boldsymbol{W}(z) = \boldsymbol{I}$. It can be verified that the controller defined by $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$ on a closed loop around $\boldsymbol{G}(z)$ does not stabilise the system. The bandwidth of $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$ can be reduced by choosing $a_1 = a_2 = \{0.7, 0.8, 0.9\}$. The closed loop is stable for the three choices of the parameters and the step responses of each design are shown in Fig. 1. These results verify the bandwidth reduction as the design parameters approach $z = 1$.

Example 5. Consider the simplified model of a high purity distillation column of eq. (12.17) in (Skogestad and Postlethwaite, 1996) and its zero order hold discrete time equivalent given by

$$\boldsymbol{G}(z) = \frac{1}{z - 0.9934} \begin{bmatrix} 0.58339 & -0.57408 \\ 0.71893 & -0.72824 \end{bmatrix}, \qquad (36)$$

where the chosen sampling time is $0.5[min]$. The PM associated to $\boldsymbol{G}(z)$ is

$$\boldsymbol{\Phi} = \begin{bmatrix} 0.198 & 0.192 \\ 0.29 & 0.31 \end{bmatrix}, \qquad (37)$$

so that using a triangular plant model is a reasonable choice. As in the previous example, the triangular model $\boldsymbol{G_t}(z)$ is the lower triangular portion of $\boldsymbol{G}(z)$ and, on first instance, we can use $a_1 = a_2 = 0$ in Theorem 3 to compute $\boldsymbol{Q}_{\boldsymbol{t}\,\boldsymbol{opt}}(z)$. It can be verified that, in this case, the

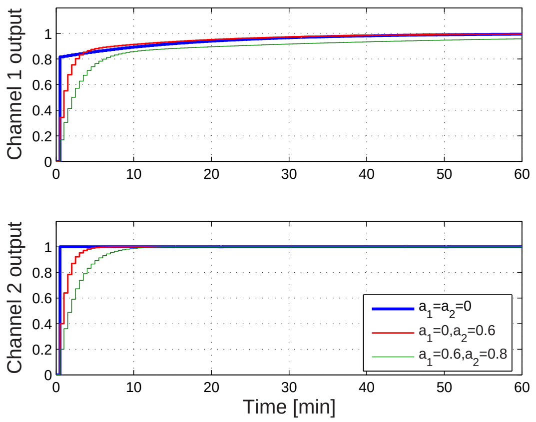

Fig. 2. Step responses for different triangular designs in Example 5.

controller defined by $\boldsymbol{Q_{t\,opt}}(z)$ on a loop around $\boldsymbol{G}(z)$ stabilises the plant. If it is needed, we can perform a fine tune on a_1 and a_2 to achieve a desired performance. Different choices for a_1 and a_2 are made and the step responses for each one are shown in Fig. 2. The results are satisfactory since we are able to modify the bandwidth of the design and hence, tune the performance to satisfy some additional transient requirements that could arise depending on the application.

7. CONCLUSIONS

This paper proposes a new methodology for the design of triangular discrete time controllers for full MIMO stable plants. The method is inherently suboptimal since the optimisation procedure is formulated considering only the triangular portion of the plant. The full MIMO nature of the plant is taken into account by adjusting the loop bandwidth through the weighting function in the cost index. The proposed method yields an explicit solution for the controller, which provides integral action.

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