Abstract: The problem of robust stabilization of a class of time-varying dynamical systems with uncertain parameters and a bounded time-varying state delay is considered. The proposed controllers can guarantee global uniform exponential stability and global uniform practical convergence of the uncertain time-varying dynamical systems with a state delay. Moreover, since the proposed controllers are completely independent of the time delay which is only assumed to be any nonnegative bounded and continuous function, the results developed in this note are applicable to a class of dynamical systems with uncertain time-delay. Finally, an illustrative example is given to demonstrate the validity of the results.

Keywords: Time-delay systems, Lyapunov function, controllers, exponential stability, practical convergence, stabilization, robustness.

1. INTRODUCTION

Nonlinear systems with time-delay constitute basic mathematical models of real phenomena such as nuclear reactors, chemical engineering systems, biological systems, and population dynamics models. They are often a source of instability and degradation in control performance in many control problems, see (1993, Hale and Lunel).

The analysis of the stability of dynamic control systems with delay and the synthesis of controllers of them are important both in theory and practice, see (1989, Cheres and al; 1994, Niculescu and al; 1992, Phoojaruenchanachai and Furuta; 1993, Shyu and Yan; 1994, Trinh and Aldeen; 2003, Gu and al; 2005, Zhang and al). Recently, improved performance have been reported by using Lyapunov-Krasovskii methods and linear matrix inequality (LMI) techniques, see (2004, Ben Hamed and Hammami; 1998, Cao and al; 1997, Li and De Souza; 2001, Oucheriah; 1996, Wu and Mizukami).

This note deals with robust stabilization of a class of uncertain time-varying systems with uncertain parameters and a bounded time-varying state delay. The parameter uncertainties are unknown but norm-bounded, and the delay is time-varying. Using the quadratic Lyapunov function for the nominal stable system as a Lyapunov candidate function for the global system, we proposed two classes of memoryless continuous state feedback controllers, i.e., linear and nonlinear, which can guarantee global uniform exponential stability and global uniform practical convergence of a dynamical systems, respectively. We give an illustrative example to demonstrate the synthesis procedure for stabilizing the nonlinear controller proposed in this note.
Notation: Throughout this note, the adopted vector norm is the Euclidean norm and the matrix norm is the corresponding induced norm. The notation $A > 0$ denote that the matrix $A$ is positive definite. $\lambda_{\text{min, max}}(A)$ denotes the minimum, maximum eigenvalue of the matrix $A$, respectively. The matrix $I_n$ denote the identity matrix of order $n$.

2. PROBLEM FORMULATION

Consider a class of uncertain time delay system of the form

$$
\dot{x}(t) = [A + \Delta A(t)]x(t) + \Delta A_d(t)x(t - h(t)) + [B + \Delta B(t)]u(t) + Bf_i(t, x(t)) + Bf_2(t, x(t - h(t))), \quad t \geq t_0
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the current value of state, $u(t) \in \mathbb{R}^m$ is the control function, $h(t)$ is the time varying delay bounded by a know bound $\bar{h}$ i.e.

$$
0 \leq h(t) \leq \bar{h}
$$

with $\bar{h}$ is a nonnegative constant and $f_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$, $i = 1, 2$, are continuous maps, which represent on control theory, the perturbed terms of the nominal systems, and in general this could result from modeling non-linear systems, disturbances, uncertainties, ...

The matrix $A, B$ are constant matrix of appropriate dimensions and the matrix $\Delta A(t), \Delta A_d(t), \Delta B(t)$ represent the system uncertainties and are continuous.

The initial condition for (1) is given by

$$
x(t) = \phi(t), \quad t \in [t_0 - \bar{h}, t_0]
$$

(2)

It is assumed that the right-hand side of (1) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition $\phi(t)$.

We also assume that the pair $(A, B)$ is controllable. Then the matrix Riccati equation

$$
A^TP + PA - 2PB^TP = -Q
$$

(3)

has a solution $P = P^T > 0$ for any $Q = Q^T > 0$.

Assumption 1. For all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ there exist continuous matrix functions $C(t), D(t), E(t)$ of appropriate dimensions such that

$$
\Delta A(t) = BC(t), \quad \Delta A_d(t) = BD(t), \quad \Delta B(t) = BE(t)
$$

(4)

Conditions (4) define the matching conditions about uncertainties and is rather standard assumption for robust control problem, see (1997, Li and De Souza; 1994, Niculescu and al). For a dynamical system with matched uncertainties, one can always design a class of stabilizing controllers. In (2004, Ben Hamed and Hammami), for example, a class of state feedback controllers is proposed such that global uniform exponential stability for the system can be guaranteed. In (1996, Wu and Mizukami), another class of continuous state feedback controllers is proposed such that ultimate uniform boundedness for system can be guaranteed.

We assume that these matched uncertainties are bounded by

$$
\|C(t)\| \leq \alpha_1, \quad \|D(t)\| \leq \alpha_2
$$

(5)

$$
\|f_1(t, x(t))\| \leq \beta_1\|x(t)\|, \quad \|f_2(t, x(t - h(t)))\| \leq \beta_2\|x(t - h(t))\|
$$

(6)

where $\alpha_i \geq 0, \beta_i > 0; i = 1, 2$, are positive constants.

Conventionally, we now introduce the following notations

$$
\eta(t) = \frac{1}{2}\lambda_{\text{min}}(E(t) + E^T(t))
$$

(7)

$$
\sigma = \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}}
$$

where $P$ is the solution of the matrix Riccati equation (3). We also suppose that

$$
\eta(t) > -1, \quad \forall t \geq t_0
$$

(8)

3. CONTINUOUS CONTROLLER GUARANTEING GLOBAL UNIFORM EXPONENTIAL STABILITY

In this section, we propose a class of continuous controllers such that global uniform exponential stability of (1)-(2) can always be guaranteed in the presence of both uncertainties. First we give the definition of global uniform exponential stability of system (1)-(2).

Definition 1. System (1)-(2) is globally uniformly exponentially stable with rate $\gamma > 0$ if there exist a positive constant $\delta > 0$ such that the following inequality holds

$$
\|x(t)\| \leq \delta\|x(t_0)\|e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0
$$

(9)

The proposed controller is given by the following form

$$
u(t, x(t)) = -\kappa(t)B^TPx(t)
$$

(10)

where the function $\kappa(t)$ is given by
Theorem 1. The state feedback controller designed by (10) is continuous and linear in the state. Therefore, the existence of the solution to (1) under controller (10) in the usual sense can be guaranteed (1977, Hale). Moreover, the linear controller given in (10) can be implemented easily in practical control problems. In addition, this controller will guarantee global uniform exponential stability of (1), and this fact is stated in the following Assumption (1999, Han and Mahdi) and (2001, Oucheriah).

Assumption 2. Assume that there exist a calculable scalar \( q > 1 \) such that the following inequality holds

\[
V(x(t - h(t))) < q^2 V(x(t))
\]

where \( V \) is a Lyapunov quadratic function given by

\[
V(x(t)) = x^T(t)Px(t)
\]

(13)

with \( P \) is the solution of the matrix Riccati equation (3). □

Since

\[
\|x(t - h(t))\|^2 \leq \sup_{t - h \leq s \leq t} \|x(s)\|^2
\]

\[
\leq \frac{1}{\lambda_{\text{min}}(P)} \sup_{t - h \leq s \leq t} V(x(s))
\]

\[
\leq \frac{q^2}{\lambda_{\text{min}}(P)} V(x(t))
\]

\[
\leq \frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)} q^2 \|x(t)\|^2
\]

one can obtain

\[
\|x(t - h(t))\| \leq q\sigma \|x(t)\|
\]

(14)

Remark 1. The state feedback controller designed by (10) is continuous and linear in the state. Therefore, the existence of the solution to (1) under controller (10) in the usual sense can be guaranteed (1977, Hale). Moreover, the linear controller given in (10) can be implemented easily in practical control problems. In addition, this controller will guarantee global uniform exponential stability of (1), and this fact is stated in the following theorem.

Theorem 1. Consider the uncertain time-delay system (1)-(2) satisfying conditions (4)-(5)-(6)-(8) for Assumption 1 and satisfying Assumption 2. Then, the closed-loop system (1)-(2)-(10) is globally uniformly exponentially stable. □

Proof We define for a nominal system (i.e. the system in the absence of both uncertainties and time delay) a positive definite function \( V(x(t)) \) given in (13).

Let \( x(t) \) be the solution of the closed-loop dynamical system (1)-(2)-(10) for any \( t \geq t_0 \), and let the function \( \dot{V}(x(t)) \) described by (13), of the nominal system be a candidate of the Lyapunov function of the closed-loop system (1)-(2)-(10). Then we can have that for any \( t \geq t_0 \)

\[
\dot{V}(x(t)) = -x^T(t)Qx(t) + 2x^T(t)PBB^T P^x(t)
\]

\[
+ 2x^T(t)PBC(t)x(t)
\]

\[
+ 2x^T(t)PBD(t)x(t - h(t))
\]

\[
- 2\kappa(t)x^T(t)PB(I_m + E(t))B^T P^x(t)
\]

\[
+ 2x^T(t)PBf_1(t, x(t))
\]

\[
+ 2x^T(t)PBf_2(t, x(t - h(t)))
\]

(15)

In view of (7), (11) and using the inequality (12), one obtains

\[
\dot{V}(x(t)) \leq -x^T(t)Qx(t) + [2 - 2\kappa(t)(1 + \eta(t))] \times
\]

\[
\|B^T P^x(t)\|^2 + 2x^T(t)P^x(t)
\]

\[
[C(t)x(t) + D(t)x(t - h(t))] + f_1(t, x(t)) + f_2(t, x(t - h(t)))
\]

\[
\leq -x^T(t)Qx(t) - \delta_1^2(\alpha_1 + \beta_1)^2
\]

\[
+ \delta_2^2 \sigma^2(\alpha_2 + \beta_2)^2
\]

\[
+ 2[(\alpha_1 + \beta_1) + q\sigma(\alpha_2 + \beta_2)] \times
\]

\[
\|B^T P^x(t)\| \|x(t)\|
\]

\[
= -x^T(t)Qx(t) - \delta_1(\alpha_1 + \beta_1) \|B^T P^x(t)\|
\]

\[
- \frac{1}{\delta_1} \|x(t)\|^2 + \frac{1}{\delta_1} \|x(t)\|^2
\]

\[
- [\delta_2 \sigma(\alpha_2 + \beta_2) \|B^T P^x(t)\| - \frac{q}{\delta_2}] \|x(t)\|^2
\]

\[
+ \frac{q^2}{\delta_2} \|x(t)\|^2
\]

So

\[
\dot{V}(x(t)) \leq -x^T(t)Qx(t) + \frac{1}{\delta_1} + \frac{q^2}{\delta_2} \|x(t)\|^2
\]

(15)

In view of the inequality (15) and let

\[
\xi := \lambda_{\text{min}}(Q) - \frac{1}{\delta_1} + \frac{q^2}{\delta_2}
\]

(16)

one obtains

\[
\dot{V}(x(t)) \leq -\xi \|x(t)\|^2
\]

(17)

If the control gain parameters \( \delta_1 \) and \( \delta_2 \) are selected such that (12) is satisfied, then a sufficiently small \( q > 1 \) exist such that \( \xi > 0 \). Since

\[
\lambda_{\text{min}}(P) \|x(t)\|^2 \leq V(x(t)) \leq \lambda_{\text{max}}(P) \|x(t)\|^2
\]

(17)

Then, one gets
\[ \dot{V}(x(t)) \leq -\frac{\xi}{\lambda_{\max}(P)} V(x(t)) \]

So, it follows that
\[ V(x(t)) \leq V(x(t_0)) e^{-\frac{\xi(t-t_0)}{\lambda_{\max}(P)}} \]

and, then
\[ \|x(t)\| \leq \sigma \|x(t_0)\| e^{-\frac{\xi(t-t_0)}{\lambda_{\max}(P)}} \]

Hence, the closed-loop system (1)-(2)-(10) is globally uniformly exponentially stable with rate \(-\xi/\lambda_{\max}(P)\) where \(\xi\) is given by (16). \(\square\)

4. CONTINUOUS CONTROLLER GUARANTEEING GLOBAL UNIFORM PRACTICAL CONVERGENCE

In this section, we propose a class of continuous controllers composed of linear and nonlinear parts such that global uniform practical convergence of system (1)-(2) can always be guaranteed in the presence of both uncertainties. Next, we give the definition of global uniform practical convergence.

**Definition 2.** System (1)-(2) is globally uniformly convergent to a ball (or globally uniformly practical convergent) with radius \(r > 0\) and with rate \(\gamma > 0\) if there exist a positive constant \(\delta > 0\) such that the following inequality holds

\[ \|x(t)\| \leq r + \delta \|x(t_0)\| e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0 \quad (18) \]

**Assumption 3.** For all \((t, x) \in \mathbb{R} \times \mathbb{R}^n\), there exist continuous matrix functions \(C(t), D(t), E(t)\) such that conditions (4) and (5) holds. In the practice the condition (6) takes the following form

\[ \|f_1(t, x(t))\| \leq \varrho_1 + \beta_1 \|x(t)\|, \]
\[ \|f_2(t, x(t-h(t)))\| \leq \varrho_2 + \beta_2 \|x(t-h(t))\| \]

where \(\varrho_1 > 0, \beta_1 > 0, i = 1, 2\), are positive constants. \(\square\)

The proposed controller is given by the following form

\[ u(t, x(t)) = u_1(t, x(t)) + u_2(t, x(t)) \quad (20) \]

where

\[ u_1(t, x(t)) = -\kappa_1(t) B^T P x(t) \]
\[ u_2(t, x(t)) = -\kappa_2(t) B^T P x(t) \]

\[ \|B^T P x(t)\| + \frac{\varepsilon}{2} \]

for a small sufficiently \(\varepsilon > 0\). The control gain functions \(\kappa_1(t)\) and \(\kappa_2(t)\) are given by

\[ \kappa_1(t) = \frac{2 + \delta_1^2(\alpha_1 + \beta_1)^2 + \delta_2^2 \sigma^2(\alpha_2 + \beta_2)^2}{2(1 + \eta(t))} \]
\[ \kappa_2(t) = \frac{\varrho_1 + \varrho_2}{1 + \eta(t)} \quad (22) \]

with \(\delta_i, i = 1, 2\), satisfying condition (12).

**Remark 2.** The controller (20) consists of two parts, \(u_1(t, x(t))\) and \(u_2(t, x(t))\). Here, \(u_1(t, x(t))\) is linear in the state, and \(u_2(t, x(t))\) is a continuous (nonlinear) controller which is used to compensate for the nonlinear parts of (1) to produce a global uniform practical convergence.

We have the following result for the continuous controller design.

**Theorem 2.** Consider the uncertain time-delay system (1)-(2) satisfying conditions (4)-(5)-(8)-(9) and Assumption 2. Then, the closed-loop system (1)-(2)-(20) is globally uniformly practical convergent with radius

\[ r(\varepsilon) := \sigma \sqrt{\frac{(\varrho_1 + \varrho_2)\varepsilon}{\xi}} \quad (23) \]

where \(\xi\) is given by (16). \(\square\)

**Proof** Let \(x(t)\) be the solution of the closed-loop dynamical system (1)-(2)-(20) for any \(t \geq t_0\). The derivation of \(V(x(t))\) described by (13), along the trajectories of the closed-loop system satisfies

\[ \dot{V}(x(t)) = -x^T(t) Q x(t) + 2x^T(t) P B B^T P x(t) \]
\[ + 2x^T(t) P B C(t) x(t) \]
\[ + 2x^T(t) P B D(t) x(t - h(t)) \]
\[ - 2\kappa_1(t) x^T(t) P B (I_m + E(t)) B^T P x(t) \]
\[ - 2\kappa_2(t) x^T(t) P B (I_m + E(t)) B^T P x(t) \]
\[ \frac{\|B^T P x(t)\| + \varepsilon}{2} \]
\[ + 2x^T(t) P B f_1(t, x(t)) \]
\[ + 2x^T(t) P B f_2(t, x(t - h(t))) \]

We apply the Proof of the Theorem 1 and using the condition (19), one obtains

\[ \dot{V}(x(t)) \leq -x^T(t) Q x(t) + \frac{1}{\delta_1^2} + \frac{q^2}{\delta_2^2} \|x(t)\|^2 \]
\[ + 2(\varrho_1 + \varrho_2) \|B^T P x(t)\| \times \]
\[ [1 - \frac{\|B^T P x(t)\|}{\|B^T P x(t)\| + \frac{\varepsilon}{2}}] \]
\[ = -\xi \|x(t)\|^2 + \frac{(\varrho_1 + \varrho_2)\|B^T P x(t)\|}{\left\|B^T P x(t)\right\| + \frac{\varepsilon}{2}} \]

Therefore, it follows from (24) and from the inequality

\[ 0 \leq \frac{ab}{a + b} \leq a, \quad \forall a, b > 0 \]
that

\[ \dot{V}(x(t)) \leq -\xi \|x(t)\|^2 + (\rho_1 + \rho_2) \varepsilon \]  

(25)

In view of the inequalities (17) and using (25), one obtains

\[ \dot{V}(x(t)) \leq -\frac{\xi}{\lambda_{\text{max}}(P)} V(x(t)) + (\rho_1 + \rho_2) \varepsilon \]

Therefore

\[ \dot{V}(x(t)) \leq \left[ V(x(t_0)) - \frac{\varepsilon \lambda_{\text{max}}(P)}{\xi} e^{-\frac{\xi(t-t_0)}{\lambda_{\text{max}}(P)}} \right] + \frac{\varepsilon \lambda_{\text{max}}(P)}{\xi} \]

\[ \leq V(x(t_0)) e^{-\frac{\xi(t-t_0)}{\lambda_{\text{max}}(P)}} + \frac{\lambda_{\text{max}}(P)(\rho_1 + \rho_2) \varepsilon}{\xi} \]

Hence

\[ \|x(t)\|^2 \leq \sigma^2 \|x(t_0)\|^2 e^{-\frac{\xi(t-t_0)}{\lambda_{\text{max}}(P)}} + (r(\varepsilon))^2 \]  

(26)

where \(r(\varepsilon)\) is given by (23).

Therefore, from (26) and from the inequality

\[ 0 \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, \quad \forall a, b > 0 \]

that

\[ \|x(t)\| \leq \sigma \|x(t_0)\| e^{-\frac{\xi(t-t_0)}{\lambda_{\text{max}}(P)}} + r(\varepsilon) \]

Hence, the closed-loop system (1)-(2)-(20) is globally uniformly practical convergent with radius \(r(\varepsilon)\) and with rate \(-\frac{\xi}{\lambda_{\text{max}}(P)}\), where \(\xi\) is given by (16). \(\Box\)

Remark 3. Notice that if \(\rho_1\) and \(\rho_2\) are equal to zero then this is the case of Section 3.

Also, if we suppose \(\rho_1 = \rho_1(t)\) and \(\rho_2 = \rho_2(t)\) with \(\rho_i(t), i = 1, 2,\) goes to zero when \(t\) goes to \(+\infty\), then all solutions converge to zero.

Remark 4. The assumption that the time-varying delay variable \(h(t)\) is restricted by \(\dot{h}(t) < 1\) is often required in many papers dealing with the stability problem of dynamical systems with time varying delay. In this note, we only assume that the delay \(h(t)\) is any nonnegative bounded and continuous function. Furthermore, the proposed controllers are completely independent of time delay. Therefore, the results developed in this note are applicable to a class of dynamical systems with some perturbed time delay.

It is worth noting that the controller proposed in (1989, Cheres and al) is dependent of the bound \(\bar{h}\) of time delay. Therefore, to employ the controller proposed in (1989, Cheres and al), it is necessary to know this bound. The controllers proposed in this note, however, are completely independent of this bound.

5. ILLUSTRATIVE EXAMPLE

Consider the uncertain time-delay system (1) with

\[ A = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Delta A(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ 0 & 0 \end{pmatrix}, \]

\[ \Delta A_d(t) = \begin{pmatrix} \cos(t) & \frac{1}{2} \sin(2t) \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

\[ \Delta B(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

\[ f_1(t, x(t)) = \frac{1}{1+t} (1 + x_1(t) + x_2(t)), \]

\[ f_2(t, x(t-h(t))) = \frac{1}{2+t} (2 + x_2(t-h(t))) \]  

(27)

where \(x(t) = (x_1(t), x_2(t)) \in \mathbb{R} \times \mathbb{R}\) and \(t \geq 0\).

From (4), (5) and (19), one obtains

\[ \alpha_1 = 1, \quad \alpha_2 = 1.414, \quad \rho_1 = \rho_2 = 1, \quad \beta_1 = 2, \quad \beta_2 = 0.5, \]

\[ 1 + \eta(t) = 1 + t \]

For a given symmetric positive definite matrix

\[ Q = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \]

with \(\lambda_{\min}(Q) = 2 > 0\), the solution of the matrix Riccati equation (3) is

\[ P = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \]

with \(\lambda_{\min}(P) = 2.382 > 0\) and \(\sigma = 1.392\).

If the control gain parameters are selected as follows

\[ \delta_1 = 1.2, \quad \delta_2 = 1 \]

then a sufficiently small \(q = 1.1 > 1\) exist such that

\[ \xi = 9.55 \times 10^{-2} > 0 \]

satisfying condition (12).

From (21) and (22), the proposed controller of the uncertain time-delay system (27) is given by

\[ u(t, x(t)) = -11.03 (3x_1(t) + x_2(t)) \]

\[ - \frac{2(3x_1(t) + x_2(t))}{(1 + t)(3|x_1(t)| + x_2(t)|)^2 + 5 \times 10^{-5}} \]  

(28)

This controller ensure the global uniform practical convergence with radius \(r = 0.636 > 0\) and with rate \(\gamma = 1.03 \times 10^{-2} > 0\).
The results of the simulations of this example is depicted in Fig. 1 and 2. In Fig. 1 the evolution of states $x_1$ and $x_2$ without controller is given, and in Fig. 2 the controller action is presented. It is shown from Fig. 2 that the closed-loop time-delay dynamical system is indeed globally uniformly practically convergent.

6. CONCLUSION

In this note, the problem of robust stabilization of a class of uncertain time-varying systems with uncertain parameters and a bounded time-varying state delay has been considered. Using the Lyapunov stability theory, we have proposed two classes of memoryless continuous state feedback controllers, i.e., linear and nonlinear, which can guarantee global uniform exponential stability and global uniform practical convergence of a dynamical systems, respectively. The quadratic Lyapunov function for the nominal stable system is used as a Lyapunov candidate function for the global system. Finally, an illustrative example is given to demonstrate the synthesis procedure for stabilizing the nonlinear controller proposed in this note. It is shown that the obtained results are feasible. Therefore, our results may be expected to have some applications to some practical control problems of systems including time-varying delay.

7. REFERENCES


