Computation of Peak Output for Inputs Restricted in $\mathcal{L}_2$ and $\mathcal{L}_\infty$ Norms Using Convex Optimization

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Abstract: Control systems design by the principle of matching gives rise to problems of evaluating the peak output. This paper proposes a practical method for computing the peak output of linear time-invariant and non-anticipative systems for a class of possible sets that are characterized with mixed bounding conditions on the two- and/or the infinity-norms of the inputs and their derivatives. The original infinite-dimensional convex optimization problem is approximated as a large-scale convex programme defined in a Euclidean space, which are associated with sparse matrices and thus can be solved efficiently in practice. The numerical results show that the method performs satisfactorily, and that using a possible set with many bounding conditions can help to reduce the design conservatism and thus yields a better match.

Keywords: peak output; the principle of matching; linear systems; convex optimization; large-scale optimization; approximation.

1. INTRODUCTION

The principle of matching [Zakian, 1991, 1996, 2005] provides a new concept in the design of control systems. The principle ensures that a system output does not exceed the prescribed bound while the system is subject to all *possible inputs* (that is, all inputs that happen or are likely to happen in practice). Accordingly, a chief design criterion is

$$|v(t,f)| \le \varepsilon \qquad \forall f \quad \forall t \in \Re, \tag{1}$$

where $v(t,f)$ denotes the value of the system output v at time t in response to a possible input f and ε is a given bound.

Clearly, condition (1) is equivalent to

$$\hat{v}(\mathcal{P}) \le \varepsilon, \tag{2}$$

where $\mathcal{P}$ denotes the possible set (that is, the set of all possible inputs) and the quantity $\hat{v}(\mathcal{P})$ defined as

$$\hat{v}(\mathcal{P}) \triangleq \sup_{f\in\mathcal{P}} \sup_{t\in\Re} |v(t,f)| \tag{3}$$

is called the peak output for the possible set $\mathcal{P}$. In contrast to (1), the design objective (2) is a practical condition for matching provided $\hat{v}(\mathcal{P})$ can be computed in practice.

Consider a linear time-invariant (LTI) and non-anticipative system whose input $f : \Re \to \Re$ and output $v : \Re \to \Re$ are related by the convolution integral

$$v(t,f) = \int_{-\infty}^{\infty} h(t-\tau) f(\tau) d\tau, \tag{4}$$

where $h : \Re \to \Re$ is the impulse response of the system. Many investigators apply various analytical optimization techniques in deriving useful formulae, and hence practical methods, for computing the peak output of system (4) for various possible sets (see, for example, Lane [1992, 1995], Rutland [1994], Zakian [1986]). From these works, one can see that for a possible set characterized by many bounding conditions, it becomes difficult to use analytical optimization to derive a practical method for computing the peak output.

In contrast, by using a numerical approximation approach, this paper proposes a practical method for computing the peak output of system (4) for the possible set $\mathcal{P}_0$ defined as

$$\begin{aligned}\mathcal{P}_0 \triangleq \big\{ f : \|f\|_2 \le M_2,\ \|\dot{f}\|_2 \le D_2, \\ \|f\|_\infty \le M_\infty,\ \|\dot{f}\|_\infty \le D_\infty \big\},\end{aligned} \tag{5}$$

where the positive constants M_2, D_2, M_∞, and D_∞ are given. It is important to note that the set $\mathcal{P}_0$ is very general and the possible sets

$$\mathcal{P}_\infty = \big\{ f : \|f\|_\infty \le M_\infty, \quad \|\dot{f}\|_\infty \le D_\infty \big\},$$

which was considered in Lane [1992], and

$$\mathcal{P}_1 = \big\{ f : \|f\|_2 \le M_2, \quad \|\dot{f}\|_2 \le D_2 \big\},$$

which was considered in Lane [1995], are special cases of $\mathcal{P}_0$. The proposed method can be used to compute $\hat{v}(\mathcal{P}_\infty)$ and $\hat{v}(\mathcal{P}_1)$ as well.

The key idea used in this paper is as follows. By truncating the associated improper integrals and by using finite-difference approximation, the original infinite-dimensional convex optimization problem is approximated as a large-scale convex optimization defined in a Euclidean space.

It is worth noting that nowadays such large-scale convex optimization problems can readily be solved by efficient

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numerical algorithms such as interior-point methods (see, for example, Boyd and Vandenberghe [2004]). In this work, for example, we use the package called SeDuMi [Sturm, 1999] to solve the arising convex optimization problem.

To demonstrate the idea and the usefulness of the proposed method, this paper considers—as a case study—the problem of computing $\hat{v}(\mathcal{P}_2)$ for the set $\mathcal{P}_2$ given by

$$\mathcal{P}_2 \triangleq \{f : \|f\|_2 \le M_2, \|\dot{f}\|_2 \le D_2, \|f\|_\infty \le M_\infty\}.$$

Now, it is worth explaining the advantage of using the set $\mathcal{P}_2$ in comparison with $\mathcal{P}_1$. Both $\mathcal{P}_1$ and $\mathcal{P}_2$ are useful in certain practical applications in which all the possible inputs are transient (see, for example, Lane [1995, 2005], Arunsawatwong [2005]). However, since $\mathcal{P}_2$ has more restrictions than $\mathcal{P}_1$, it can be seen that $\mathcal{P}_2$ contains less fictitious inputs. Hence, in accordance with the principle of matching, a better match can be obtained.

The organization of this paper is as follows. Section 2 gives a general formulation of the problem of evaluating the peak output $\hat{v}(\mathcal{P})$ for a given $\mathcal{P}$, and explains the finite-difference schemes used in approximating integrals and derivatives. Section 3 provides the details of the proposed numerical procedure and also shows that the resultant optimization problem is convex independent of the value of the difference used in the approximation. In Section 4, numerical examples are given by reconsidering the computation of the peak outputs of the wind turbine system given in Lane [1995, 2005]. Specifically, the peak outputs $\hat{v}(\mathcal{P}_1)$ are obtained using the proposed method and are compared with those obtained by using Lane's [1995] method; the peak outputs for $\hat{v}(\mathcal{P}_2)$ are also computed and discussed. Finally, the conclusions and discussion are provided in Section 5.

2. PROBLEM FORMULATION

Let $\mathcal{P}$ be one of the possible sets considered in this paper. We can see that if f belongs to a possible set $\mathcal{P}$, then the time shifted input also belongs to the possible set $\mathcal{P}$. The peak output $\hat{v}(\mathcal{P})$ for system (4) can be determined by considering just one particular time t. In this connection, we assume that the peak output occurs at $t = 0$. Moreover, $-f \in \mathcal{P}$. See Lane [1992] for more details of this. Hence, the problem of computing $\hat{v}(\mathcal{P})$ for system (4) becomes

$$\hat{v}(\mathcal{P}) = \sup\{\mathrm{I}(f) : f \in \mathcal{P}\}, \tag{6}$$

where $\mathrm{I}(f)$ is the cost function given by

$$\mathrm{I}(f) = \int_{-\infty}^{\infty} h(-\tau)f(\tau)\,d\tau. \tag{7}$$

The impulse response considered in this paper is of the form

$$h(t) = \beta\delta(t) + h_1(t), \tag{8}$$

where β is a real number, δ denotes Dirac delta function and $h_1 : \Re \mapsto \Re$ is a bounded and piecewise continuous function.

This section presents the methodology that is used in approximating the infinite-dimensional convex optimization (6) as a convex optimization problem defined in a Euclidean space, which is more easily tractable.

2.1 Cost function approximation

The improper integral in (7) can be expressed as

$$\mathrm{I}(f) = \beta f(0) + \int_{-T}^{T} h_1(-\tau)f(\tau)\,d\tau + e(f,T), \tag{9}$$

where

$$e(f,T) = \int_{-\infty}^{-T} h_1(-\tau)f(\tau)d\tau + \int_{T}^{\infty} h_1(-\tau)f(\tau)d\tau. \tag{10}$$

Notice that $e(f,T)$ is the error functional in connection with the truncation of the indefinite integral in (7).

The following proposition shows that for a bounded-input bounded-output (BIBO) stable system, the error functional $e(f,T)$ converges to zero as $T \to \infty$.

Proposition 1. If system (4) is BIBO stable and if the possible input f is bounded on the interval $(\infty, -T] \cup [T, \infty)$, then $e(f,T)$ defined in (10) converges to zero as $T \to \infty$.

Proof It is sufficient to consider only one term on the right hand side of (10). Letting $t = -\tau$ yields

$$\int_{-\infty}^{-T} h_1(-\tau)f(\tau)\,d\tau \;\le\; F_m \int_{T}^{\infty} |h_1(t)|\,dt, \tag{11}$$

where $F_m = \{\sup|f(t)| : t \le -T\}$. For a BIBO stable system, the integral term on the left hand side of (11) converges to zero as $T \to \infty$ (see, for example, Chen [1983]). Hence, the proof is complete. □

Clearly, Proposition 1 shows that for a sufficiently large $T > 0$, the improper integral in (7) can be well approximated as

$$\mathrm{I}(f) \approx \beta f(0) + \int_{-T}^{T} h_1(-\tau)f(\tau)\,d\tau. \tag{12}$$

From (12), one can easily approximate $\mathrm{I}(f)$ using finite-differences.

To this end, for $t \in [-T, T]$, the trajectories $h_1(t)$ and $f(t)$ are represented by vectors $\bar{h}$, $\bar{f}_0 \in \Re^{2n+1}$ such that

$$\bar{h} \triangleq [h_{-n},\ h_{-n+1}, \ldots, h_{n-1},\ h_n]^{\mathsf{T}} \tag{13}$$

and

$$\bar{f}_0 \triangleq [f_{-n},\ f_{-n+1}, \ldots, f_{n-1},\ f_n]^{\mathsf{T}}, \tag{14}$$

where $h_i = h_1(t_i)$ and $f_i = f(t_i)$, respectively. The time point t_i is given by

$$t_{-n} = -T,$$
$$t_{i+1} = t_i + \sigma \quad \text{for} \quad i = -n, -n+1, \ldots, n-1,$$

where the uniform difference $\sigma = T/n$.

Now it is important to explain how to set the value of f_{-n} in our formulation. Lane [1992] has pointed out that it is necessary to set the initial value of f to be zero (that is, $f_{-n} = 0$) so that the worst input is unique. Thus, $\bar{f}_0$ in (14) is replaced by

$$\bar{f} \triangleq [f_{-n+1},\ f_{-n+2}, \ldots, f_{n-1},\ f_n]^{\mathsf{T}} \in \Re^{2n}. \tag{15}$$

Next, we use Simpson's rule (see, for example, Kincaid and Cheney [2001]) in approximating the truncated integral in (12). If the positive number n is chosen to be even, then we have

$$\mathrm{I}(f) \;\approx\; c_h^{\mathsf{T}}\bar{f} \;\triangleq\; \mathbb{I}(\bar{f}), \tag{16}$$

where the vector $c_h \in \Re^{2n}$ in (16) is given by

$$c_h = \tfrac{\sigma}{3}[4h_{n-1},\ 2h_{n-2}, \ldots, 4h_{-1},\ (\tfrac{3}{\sigma})\beta + h_0,\ 0_{1\times n}]^{\mathsf{T}}. \tag{17}$$

The zero subvector in (17) is a consequence of the fact that system (4) is non-anticipative (that is, $h(t) = 0 \ \forall t < 0$).

Hence, the cost $\mathrm{I}(f)$ in (7) is approximated by $\mathbb{I}(\bar{f})$ defined in (16).

2.2 Constraints approximation

Derivative approximation When the trajectory $f(t)$ is represented by the vector $\bar{f}$ in (15), it is readily appreciated that the trajectory $\dot{f}(t)$ can also be represented by a vector $\bar{f}_d$ such that

$$\bar{f}_d = Q_d \bar{f}, \tag{18}$$

where Q_d and the dimension of $\bar{f}_d$ depend on the formula used in approximating the derivative.

Suppose the first-order forward difference formula

$$\dot{f}(t_i) \approx \frac{f(t_i + \sigma) - f(t_i)}{\sigma} \tag{19}$$

is used (see, for example, Kincaid and Cheney [2001]). Then, the matrix $Q_d \in \Re^{2n \times 2n}$ in (18) becomes

$$Q_d = \frac{1}{\sigma}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

and $\bar{f}_d \in \Re^{2n}$. Notice that, in this case, Q_d has a full column rank.

Restriction in two-norms Following the method used in Section 2.1, it is readily appreciated that the two-norm of function f is approximated as

$$\|f\|_2^2 \approx \sum_{k=1}^{2n-1} q_k f_k^2 = \bar{f}^{\top} Q_1 \bar{f}, \tag{20}$$

where f_k is the kth element of $\bar{f}$ and $Q_1 \in \Re^{2n \times 2n}$ is a diagonal matrix whose diagonal elements $Q_{kk} = q_k$. When Simpson's rule is applied, it is easy to verify that

$$q_k = \begin{cases} \frac{\sigma}{3}\left(3 + (-1)^{k+1}\right) & \text{for } k = 1, 2, \dots, 2n-1 \\ \frac{\sigma}{3} & \text{for } k = 2n. \end{cases}$$

Consequently, Q_1 is positive definite.

Similarly, the two-norm of $\dot{f}$ is approximated as

$$\|\dot{f}\|_2^2 \approx \bar{f}_d^{\top} Q_2' \bar{f}_d = \bar{f}^{\top} Q_2 \bar{f}, \tag{21}$$

where $Q_2' \in \Re^{2n \times 2n}$ is diagonal and positive definite. Next we will show that Q_2 in (21) is positive definite.

Proposition 2. Assume that $U \in \Re^{n \times n}$ is positive definite and $Z \in \Re^{n \times m}$ has a full column rank. Then $P = Z^{\top} U Z$ is also positive definite.

Proof Let $y = Zx$. Since $Z = [z_1, z_2, \dots, z_m]$ has a full column rank, its columns z_i $(i = 1, 2, \dots, m)$ are linearly independent. Hence, $y = 0$ iff $x = 0$. Since U is positive definite, $y^{\top} U y = x^{\top} P x = 0$ iff $x = 0$. Hence, the proof is complete. □

From Proposition 2, it follows that when the first-order forward difference formula is used, Q_2 is positive definite for any $\sigma > 0$.

From the above, the bounding conditions $\|f\|_2 \le M_2$ and $\|\dot{f}\|_2 \le D_2$ are replaced by

$$\bar{f}^{\top} Q_1 \bar{f} \le M_2^2 \tag{22}$$

and

$$\bar{f}_d^{\top} Q_2 \bar{f}_d \le D_2^2, \tag{23}$$

respectively.

Restriction in infinity-norms The inequalities $\|f\|_\infty \le M_\infty$ and $\|\dot{f}\|_\infty \le D_\infty$ can be replaced by

$$-M_\infty \le f_i \le M_\infty \quad \text{for } i = -n+1, -n+2, \dots, n \tag{24}$$

and

$$-D_\infty \le f_{d_i} \le D_\infty \quad \text{for } i = -n+1, -n+2, \dots, n, \tag{25}$$

where f_i and f_{d_i} are the ith element of $\bar{f}$ and $\bar{f}_d$, respectively.

Let $x \preceq y$ denote componentwise inequality between vectors x and y. Inequalities (24) can be rewritten as

$$\mathbf{I}\bar{f} \preceq M_\infty \bar{\mathbf{1}} \quad \text{and} \quad -\mathbf{I}\bar{f} \preceq M_\infty \bar{\mathbf{1}}, \tag{26}$$

where $\mathbf{I}$ and $\bar{\mathbf{1}}$ denote the identity matrix and a vector with all components being one, respectively. Similarly, (25) can be rewritten as

$$Q_d \bar{f} \preceq D_\infty \bar{\mathbf{1}} \quad \text{and} \quad -Q_d \bar{f} \preceq D_\infty \bar{\mathbf{1}}. \tag{27}$$

3. THE PROPOSED METHOD

The approximated constraints are recapped as follows.

- $\|f\|_2 \le M_2$ is replaced by
$$\bar{f} Q_1 \bar{f} \le M_2^2 \tag{28}$$
- $\|\dot{f}\|_2 \le D_2$ is replaced by
$$\bar{f} Q_2 \bar{f} \le D_2^2 \tag{29}$$
- $\|f\|_\infty \le M_\infty$ is replaced by
$$\left.\begin{aligned} \mathbf{I}\bar{f} &\preceq M_\infty \bar{\mathbf{1}} \\ -\mathbf{I}\bar{f} &\preceq M_\infty \bar{\mathbf{1}} \end{aligned}\right\} \tag{30}$$
- $\|\dot{f}\|_\infty \le D_\infty$ is replaced by
$$\left.\begin{aligned} Q_d\bar{f} &\preceq D_\infty \bar{\mathbf{1}} \\ -Q_d\bar{f} &\preceq D_\infty \bar{\mathbf{1}} \end{aligned}\right\} \tag{31}$$

The problem of computing $\hat{v}(\mathcal{P}_0)$ is the solution of the following optimization problem

$$\max\left\{ \mathbb{I}(\bar{f}) : \bar{f} \text{ satisfies } (28), (29), (30) \text{ and } (31) \right\}, \tag{32}$$

where $\mathbb{I}(\bar{f})$ is given in (16).

Problem (32) is a generalization of many cases. For example, when the bounds M_∞ and D_∞ become infinite, $\mathcal{P}_0$ becomes $\mathcal{P}_1$. In this case, only constraints (28) and (29) are used in (32) for computing $\hat{v}(\mathcal{P}_1)$. Similarly, for computing $\hat{v}(\mathcal{P}_2)$, inequality (31) is removed.

Proposition 3. When both Simpson's rule and the first-order forward difference formula (19) are used, the optimization problem in (32) is convex for any difference $\sigma > 0$.

Proof For $\sigma > 0$, it follows from (28) and (29) that Q_1 is positive definite and, by Proposition 2, so is Q_2. Clearly, the cost function in (32) is linear in $\bar{f}$. As a consequence, it follows from (30)–(31) that (32) is a convex optimization problem. □

A standard form of convex optimization problem can be found in, for example, Boyd and Vandenberghe [2004]. It is noted that the method used in numerical differentiation and integration is open for system engineers as long as the resultant optimization problem in (32) is convex.

The procedure for computing the peak output for system (4) for the possible set $\mathcal{P}_0$ and its special cases is summarized as follows.

Step 1: Determine the impulse response h of the system.
Step 2: Construct the vector c_h defined in (17).
Step 3: For the derivative of inputs, form a full column rank matrix Q_d.
Step 4: Formulate the two-norm restrictions (if any) as quadratic inequalities (28) and/or (29).
Step 5: Formulate the infinity-norm restrictions (if any) as linear inequalities, (30) and/or (31).
Step 6: Solve the resultant convex optimization problem.

Clearly, for BIBO stable systems defined in (4), if the time T and the number of points n are chosen to be sufficiently large, then the optimization problem (6) can be approximated well by the large-scale convex optimization problem (32). Notice that the matrices associated with (32) are sparse.

With the current advance in interior-point methods, large-scale convex optimization problems with more than 100,000 variables can usually be solved in practice within a reasonable amount of time. Throughout this work, SeDuMi package [Sturm, 1999] is employed in solving (32) and has proved to be efficient. In addition, other optimization solver packages may also be used, for example, SDPT3 [Tütüncü et al., 2003].

4. NUMERICAL EXAMPLES

In this section, the proposed method is used for computing $\hat{v}(\mathcal{P}_1)$ and $\hat{v}(\mathcal{P}_2)$ for the wind turbine control system given in Lane [2005]. For convenience, the system data are provided in Appendix B. For the purpose of this section, the meanings of the system variables are omitted.

All the computations given here are obtained by using MATLAB 7.2 for Windows XP on laptop PC with Intel Pentium(R) processor 1.60GHz and 512MB RAM.

For the purpose of comparison, we compute the peak outputs $\hat{v}(\mathcal{P}_1)$ of output ports 1–5 using the proposed method and Lane's [1995] method, which was derived from an analytical optimization technique. To this end, let $\hat{v}_L(\mathcal{P}_1)$ and $\hat{v}_P(\mathcal{P}_1)$ denote the values of $\hat{v}(\mathcal{P}_1)$ obtained by Lane's method and the proposed method, respectively. The peak outputs $\hat{v}_L(\mathcal{P}_1)$ and their relative errors defined by

$$E_r \triangleq \frac{|\hat{v}_L(\mathcal{P}_1) - \hat{v}_P(\mathcal{P}_1)|}{\hat{v}_L(\mathcal{P}_1)} \times 100\% \qquad (33)$$

are computed and tabulated in Table 1. Note that in this case, $T = 160$ and $n = 10,000$ are used. It can be seen that the proposed method gives satisfactory results.

Table 1. Comparison between $\hat{v}_L(\mathcal{P}_1)$ and $\hat{v}_P(\mathcal{P}_1)$.

Output port	$\hat{v}_L(\mathcal{P}_1)$	$E_r\%$
1	1.1834	9.66×10^{-3}
2	0.0545	9.73×10^{-3}
3	0.3732	2.90×10^{-2}
4	0.1186	6.65×10^{-3}
5	0.1292	3.70×10^{-3}

Next, we suppose that the bound M_∞ is known or specified. Consider the cases of $M_\infty = 5$, $M_\infty = 4$ and $M_\infty = 1.7$, respectively. The peak outputs $\hat{v}(\mathcal{P}_2)$ are computed using the proposed method and then compared with $\hat{v}(\mathcal{P}_1)$. To this end, let $f^*(\mathcal{P})$ denote the worst input for the set $\mathcal{P}$ and let $\mathcal{P}_{2,M_\infty}$ denote the possible set $\mathcal{P}_2$ with the infinity-norm bound M_∞, respectively.

In Table 2, the results show that if $M_\infty \geq \|f^*(\mathcal{P}_1)\|_\infty$, then $\hat{v}(\mathcal{P}_1) = \hat{v}(\mathcal{P}_2)$. Consider output port 1, for example, $\|f^*(\mathcal{P}_1)\|_\infty = 4.4747$ implies that $f^*(\mathcal{P}_1) \in \mathcal{P}_{2,5}$. Accordingly, $\hat{v}(\mathcal{P}_1) = \hat{v}(\mathcal{P}_{2,5})$. Since $f^*(\mathcal{P}_1) \notin \mathcal{P}_{2,4}$ and $\mathcal{P}_{2,4} \supset \mathcal{P}_{2,1.7}$, it follows that $\hat{v}(\mathcal{P}_{2,1.7}) < \hat{v}(\mathcal{P}_{2,4}) < \hat{v}(\mathcal{P}_1)$. These results agree with the fact given in Appendix A.

The plots of the approximated worst inputs of output port 1 for $\mathcal{P}_1$, $\mathcal{P}_{2,5}$, $\mathcal{P}_{2,4}$ and $\mathcal{P}_{2,1.7}$ are illustrated in Figures 1a–1d, respectively. In Figures 1-a and 1-b, the plots show no difference. However, when $\mathcal{P}_{2,4}$ and $\mathcal{P}_{2,1.7}$ are used, one can see from the plots that the $f^*(\mathcal{P}_{2,4})$ and $f^*(\mathcal{P}_{2,1.7})$ reach their magnitude bounds.

In Figure 2, the computational time used versus the number of variables $2n$ and the relative error in the peak output $\hat{v}(\mathcal{P}_1)$ of output port 1 versus $2n$ are given. Figure 2 shows a trade-off between the accuracy and the computational time. It can be seen that for all values of $2n$, the computational time is less than 8 seconds.

5. CONCLUSIONS AND DISCUSSION

In this paper, we have developed a practical method for computing the peak output of system (4) for the possible set $\mathcal{P}_0$ (see (5)) using a numerical approximation approach. In this case, one can clearly see that with many bounding conditions it becomes very difficult to employ analytical optimization to derive a formula (and hence a method) for computing the peak output.

The original problem of computing the peak output is an infinite-dimensional convex optimization problem. By using the numerical approach, the problem is approximated as a large-scale convex programme defined in a Euclidean space, which can be solved in practice by an efficient convex programme solver with powerful computing facilities.

In view of the possible set $\mathcal{P}_0$, which is general, and the numerical approach developed in this paper, system engineers can impose many bounding conditions (for example, the set $\mathcal{P}_2$) in order to characterize a possible set with an appropriate model. Hence, in connection with the principle of matching, a better match between the system and the environment can be achieved.

Table 2. The peak outputs $\hat{v}(\mathcal{P}_1)$ and $\hat{v}(\mathcal{P}_2)$ for $M_\infty = 5, 4$, and 1.7.

Output port	$\hat{v}_L(\mathcal{P}_1)$	$\|f^*(\mathcal{P}_1)\|_\infty$	$\hat{v}(\mathcal{P}_{2,5})$	$\hat{v}(\mathcal{P}_{2,4})$	$\hat{v}(\mathcal{P}_{2,1.7})$
1	1.1834	4.4747	1.1834	1.1770	0.8236
2	0.0545	4.4670	0.0545	0.0542	0.0422
3	0.3732	2.7708	0.3732	0.3732	0.3656
4	0.1186	5.6243	0.1152	0.1024	0.0582
5	0.1292	2.0072	0.1292	0.1292	0.1288

Since we use the convolution integral representation (4) in our formulation, the proposed method is applicable to lumped- and distributed-parameter systems as long as their impulse responses are obtainable. Therefore, the method is usefully applicable to a wide class of system (4).

It is interesting to note that the methodology adopted in this paper is useful in determining the peak output for other possible sets, for example, the possible sets considered in Rutland [1994].

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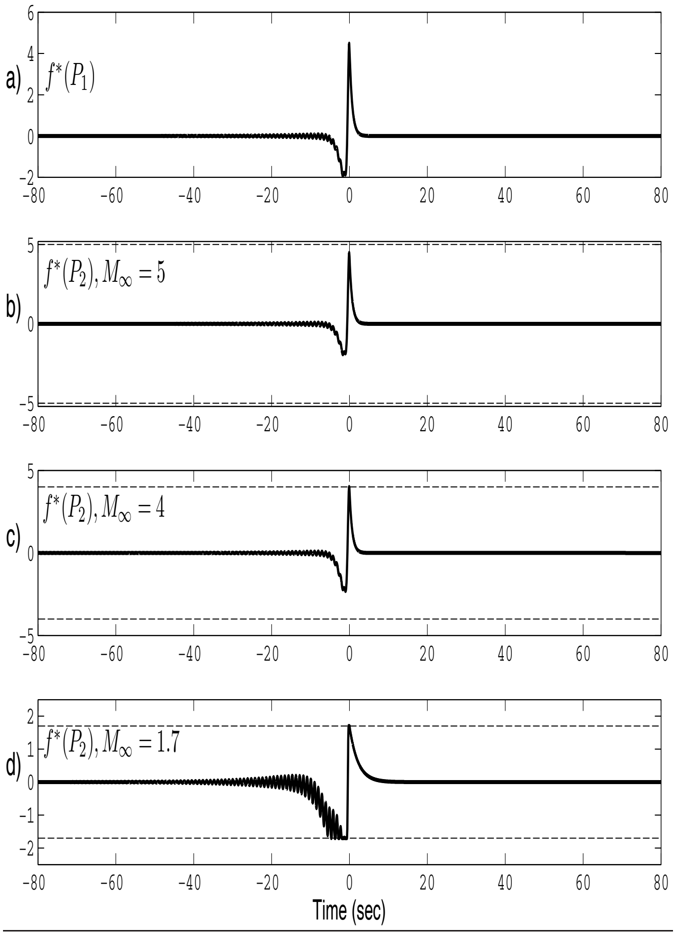

Fig. 1. Approximately worst inputs for: a) $\mathcal{P}_1$, b) $\mathcal{P}_2$ with $M_\infty = 5$, c) $\mathcal{P}_2$ with $M_\infty = 4$, d) $\mathcal{P}_2$ with $M_\infty = 1.7$.

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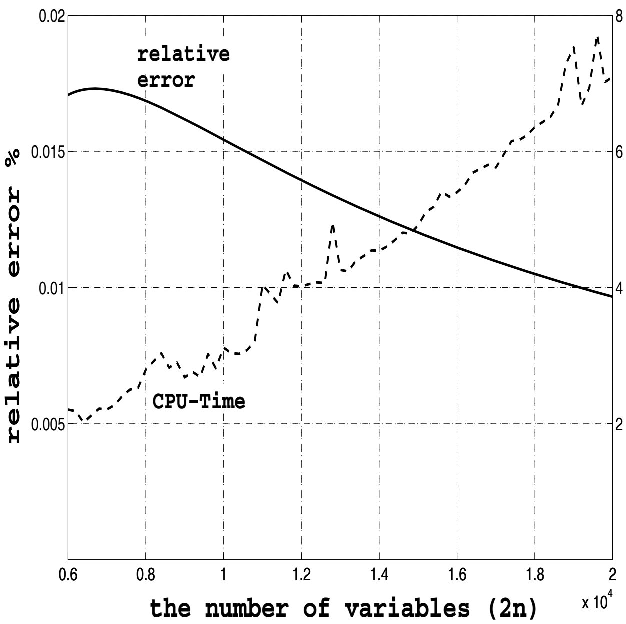

Fig. 2. Relative error and computational time versus the number of variables.

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APPENDIX A

Proposition A.1 Given two possible set $\mathcal{P}_a$ and $\mathcal{P}_b$. If $\mathcal{P}_a \subset \mathcal{P}_b$, then $\hat{v}(\mathcal{P}_a) \leq \hat{v}(\mathcal{P}_b)$.

Proof Suppose that w^* and f^* are the inputs which produce the peak outputs for $\mathcal{P}_a$ and $\mathcal{P}_b$, respectively. Since $\mathcal{P}_a \subset \mathcal{P}_b$, it is clear that $w^* \in \mathcal{P}_b$. By the definition of the peak output, it readily follows that

$$\hat{v}(\mathcal{P}_a) \triangleq \sup_t \{v(t, w^*)\} \leq \sup_t \{v(t, f^*)\} \triangleq \hat{v}(\mathcal{P}_b)$$

□

APPENDIX B

The state equation of the wind turbine system given in Lane [2005] is

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bf(t) \\ v(t) &= Cx(t) + Df(t) \end{aligned}\right\}, \tag{34}$$

where x, f and v denote the state vector, the system input and the system output vector, respectively. The nonzero elements of the matrices A, B, C and D are as follows.

$$\begin{aligned}
&A_{2,1} = -7.8083, && A_{4,1} = -27.5330, && A_{5,1} = 20.1590, \\
&A_{1,2} = 1.0000, && A_{2,2} = -3.1912, && A_{4,2} = -10.9420, \\
&A_{5,2} = 7.8540, && A_{4,3} = -4.4762, && A_{2,4} = -0.0549, \\
&A_{3,4} = 1.0000, && A_{4,4} = -0.2286, && A_{2,5} = 0.5840, \\
&A_{4,5} = 2.5531, && A_{5,5} = -3.4843, && A_{6,5} = 0.2500, \\
&A_{2,6} = -0.7734, && A_{4,6} = -3.3808.
\end{aligned}$$

$$B_{2,1} = 0.0471, \quad B_{4,1} = 0.1429.$$

$$\begin{aligned}
&C_{1,1} = 20.1590, && C_{2,1} = 1.0000, && C_{4,1} = -3.2122, \\
&C_{5,1} = 15.7770, && C_{1,2} = 7.8540, && C_{4,2} = -1.2515, \\
&C_{5,2} = 3.1210, && C_{3,3} = 1.0000, && C_{5,4} = 0.0687, \\
&C_{4,5} = 0.2979, && C_{5,5} = -1.8674, && C_{4,6} = -0.3944, \\
&C_{5,6} = 0.9679.
\end{aligned}$$

$$D_{5,1} = -0.0589.$$

The parameters $M_2 = 7.6$ and $D_2 = 4.8$ are used.