Cheap Computation of Optimal Reduced Models Using Symbolic Computation

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Abstract: Many methods have been proposed for the reduction of single-variable and multi-variable systems having ordinary denominators and those having delays in their numerators and denominators. In this paper, an algorithm for this purpose is proposed that can be easily used by anybody with not too advanced knowledge of mathematics yielding optimal reduced models. This algorithm makes use of the symbolic capabilities of Computer Algebraic Systems (CAS) like Mathematica, MATLAB and Maple to carry out the model reduction. An advantage of the algorithm is that it can be easily automated and used for rational, irrational, retarded SISO and MIMO systems.

1. INTRODUCTION

The host of methods employed to derive simpler models from a high order one are referred to as Model Order Reduction (MOR) methods. MOR tries to quickly capture the essential features of a high order system and still retain its essential characteristics. In the algorithm proposed, the performance index used is the time-weighted integral of squared-error (ISE) between the responses of the original and reduced models. The input used is a combination of impulse and step thus ensuring both transient and steady state accuracy (this input is persistent enough as shown by the agreement in the frequency responses of the original and reduced models). Here, the whole calculation is done in the time domain unlike previous efforts (for example Taiwo, 1999 and Xue and Atherton, 1994) where computation was carried out in the frequency domain.

It is to be noted that the focus of this paper is computational details. The theory has been adequately tackled in Taiwo's 1999 paper. The results obtained by the application of these algorithms agree with those obtained earlier although this technique is more user-friendly.

2. PROPOSED ALGORITHM

For this work, the input used is of the form

$$u(s) = u_0 + \frac{u_1}{s}$$

The main steps involved in the algorithm are:

1. Obtain the transfer function of the original system, $G(s)$, the weighting function, $h(t)$, required, the input, $u(t)$ to the plant and the form of the reduced model desired, $G_r(s)$.
2. Find the output of the original and reduced models, $y(t)$ and $y_r(t)$. This involves multiplying the respective transfer functions by the input and finding the inverse Laplace transform of the resulting functions i.e.
   $$y(s) = G(s)u(s)\,,\ y(t) = \mathcal{L}^{-1}[y(s)]$$
3. Find the error function which is the difference between the two functions found in step 2 above
   $$e(t) = y(t) - y_r(t)$$
4. For zero steady state error between the original and reduced models, set $e(\infty) = 0$. This can be found by applying the final value theorem. This is then used to get one of the parameters of the reduced model in terms of the others and hence reduce the dimension of the optimisation problem to be solved later.
5. The cost function is given by
   $$J = \int_0^\infty h(t)e^2(t)dt \qquad (1)$$
   This is evaluated using the symbolic capabilities of the Computer Algebraic System (CAS) in use. The following should be noted while finding the cost function:
   i. Most CASs cannot handle ∞ and will give an error message when it is encountered in a computation. But the expression for J has ∞ in it! Here ∞ is replaced by a very large number such as a certain multiple of the original model settling time (for example 3 times).
   ii. If the original or reduced system model have delay(s) in the numerator, it is better to break the integral into intervals around the delay. For instance,

if there is a delay at time τ in any of the models then Eq. (1) becomes

$$J = \int_0^{\tau} h(t)e_1^2(t)dt + \int_{\tau}^{\infty} h(t)e_2^2(t)dt \qquad (2)$$

where $e_1(t)$ and $e_2(t)$ are obtained by substituting 0 or 1, respectively, for any terms $1(t-\tau)$ in $e(t)$ depending on whether $t > \tau$ or $t \le \tau$. This makes the computation of the integral faster.

iii. To remove the decaying part of J that will still remain due to approximation of the infinity by a large number, the expression in J above is then Chopped (use the function Chop in Mathematica or any other suitable functions in MATLAB or Maple) to discard the remaining decaying part less than a set tolerance (for example 10^{-10}).

6. The expression of J obtained in step 5 above is then minimized with respect to the parameters of the reduced model. While minimizing, constraints could be put such that the reduced model is stable (the poles have negative real parts). This is easily implemented while using the NMinimize function in Mathematica. It should be noted that numerical methods are faster, than algebraic methods, for minimizing most of the expressions that will be obtained in step 5. At times, algebraic method may not even produce any answer till the computer runs out of computational memory!
7. The steps outlined above work perfectly well for rational systems and those with delay(s) in the numerator. For systems with delay in the denominator, the first thing to be done is finding a way to remove the delays in the denominator without altering the properties of the system significantly. This can be done in a number of ways:
   i. Using a Taylor's series expansion to approximate the delay terms
   ii. Using Pade approximant to approximate the delay terms
   iii. Rearranging the expressions to facilitate evaluation (Marshall et. al., 1992 and Taiwo et. al., 2001)

After resorting to any of (i)-(iii) above, then the steps outlined in 1-6 can be followed to obtain the parameters of the reduced model.

8. For MIMO systems the algorithm can still be applied but with some modifications. For an n by n MIMO system, the cost function is found for each element of the matrix and summed together as in:

$$J = \int_0^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(t)e_{ij}^2(t)dt \qquad (3)$$

Then steps 2 to step 7 can be applied to obtain the optimal reduced-model parameters.

9. The advantage of the algorithm is that almost the same procedure used for SISO systems could be used for MIMO systems. To reduce the time taken for the computation of the integral in Eq. (3), it is best to evaluate the integral for each element of the system and then add the resultant expressions rather than lump everything together as in Eq. (3). This is because the more complex the expression the computer has to process the more the virtual memory it will need for the operation and the higher the tendency that the computer may hang or run out of computational memory. Eq. (3) can then be written as

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^{\infty} h_{ij}(t)e_{ij}^2(t)dt \qquad (4)$$

10. In order to constrain the reduced model frequency response to have certain characteristics like the original model frequency response, the relationship between the reduced and original models parameters that will facilitate the reduced and original models frequency responses have certain similar characteristic (e.g. the same cross-over frequency) is obtained. Such equations can then be used as constraints during the optimization step of the algorithm or solved to obtain one of the parameters in terms of the others thus further reducing the dimension of the optimization problem.
11. Because a numerical scheme is used for minimizing the cost function obtained, care should be taken so that the parameters obtained correspond to the case of global minimum of the cost function and not a local minimum. What is done is to use different numerical schemes and different starting values for the parameters while minimizing until one is sure that the point obtained is a global minimum. Numerical schemes that may be used in Mathematica are NMinimize (with options NelderMead, SimulatedAnnealing, DifferentialEvolution or RandomSearch), FindMinimum (with options Newton, QuasiNewton, Gradient or ConjugateGradient) or one could write a subprogram to do the minimization.

3. RESULTS

3.1 Example 1

$$G(s) = \frac{1}{(s/80 + 1)^{80}}$$

The system (Hwang and Chuang, 1994) has a cross-over frequency of $\omega_{pco} = 3.14321$ and amplitude ratio of $|G(\omega_{pco})| = 0.940164$.

A first-order plus timed delay reduced-models of the form

$$G_r(s) = \frac{c_0 e^{-\tau \cdot s}}{c_0 + s}$$

is proposed for the model. The optimal parameters of the reduced-model obtained, together with the ISE, the cross-over frequency and the amplitude ratio at cross-over frequency, are given in Table 1. The input used is unit step. The table also contains the parameters of $G_r(s)$ computed such that the cross-over frequencies as well as the amplitude ratios of the original and reduced models are equal. Results obtained for combination of inputs are given in Table 2. The frequency and open-loop step responses of the various reduced models are shown in Fig. 1 and Fig. 2 respectively.

Table 1. Comparison of first order plus time delay reduced models

S/N	Method	c_0
1	This work for $h(t)=1$	7.53789
2	This work for $h(t)=t^2$	8.07211
3	This work for equal ω_{pco} and $\lvert G(\omega_{pco})\rvert = \lvert G_r(\omega_{pco})\rvert$	8.673144
4	Hwang and Chuang, $h(t)=1$	4.74739

Table 1. Comparison of first order plus time delay reduced models (cont'd)

S/N	τ	$\lvert G(\omega_{pco})\rvert$	ω_{pco}	ISE
1	0.884291	3.11014	0.924406	0.0017385
2	0.891091	3.11255	0.93304	0.0017994
3	0.888871	3.14321	0.940164	0.0021581
4	0.873909	2.95737	0.848781	0.0356262

Table 2. Comparison of first order plus time delay reduced models for combination of inputs

S/N	Method	u_0	u_1	c_0
1	$h(t)=1$	0.25	0.75	4.74739
2	$h(t)=t$	0.25	0.75	4.90689
3	$h(t)=t^2$	0.25	0.75	5.06638

Table 2. Comparison of first order plus time delay reduced models for combination of inputs (cont'd)

S/N	τ	$\lvert G(\omega_{pco})\rvert$	ω_{pco}	ISE
1	0.873909	2.95737	0.848781	0.0356262
2	0.877219	2.96217	0.856101	0.0357496
3	0.880514	2.96634	0.862966	0.0361137

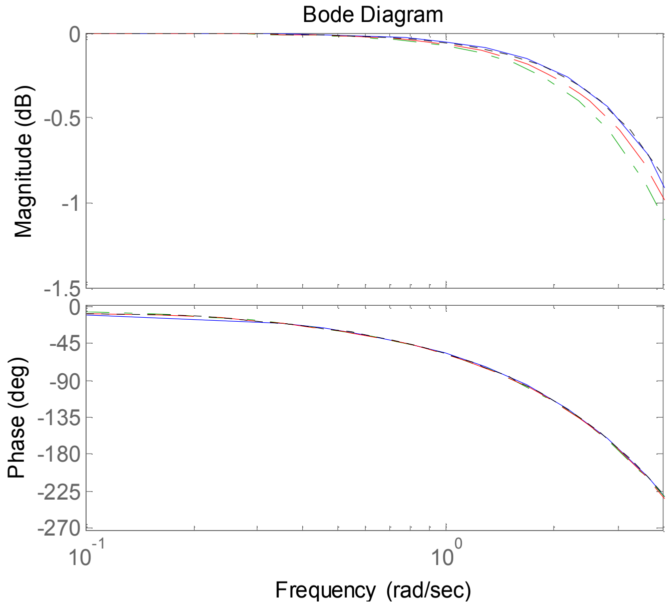

Fig. 1. Bode Plots of the original and reduced models for example 1; ——— original model, — · — reduced with h(t)=1, — — — reduced with h(t)=t^2, and · · · · reduced with equal w_{pco} and $\lvert G_{wpco}\rvert$ as the original.

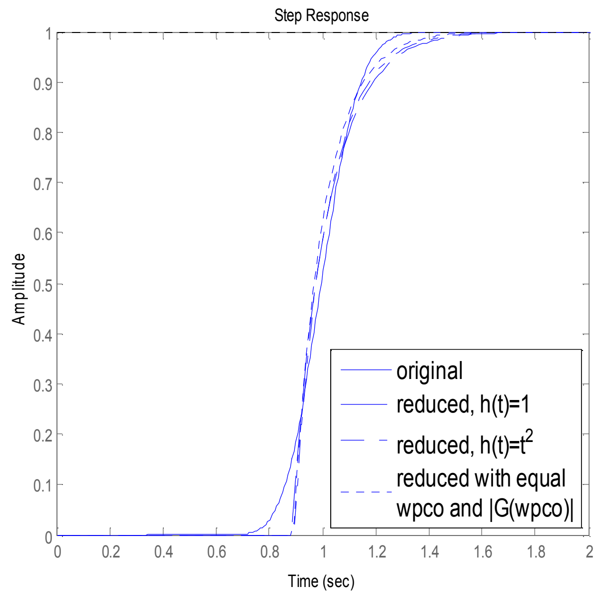

Fig. 2. Open-loop step responses of the original and reduced models for example 1

3.2 Example 2

The original model transfer function (Lepschy et. al., 1986) is

$$G(s) = \frac{1}{1 + s + 0.9s^2 + 0.44s^3 + 0.14s^4 + 0.035s^5}$$

It has a $\omega_{pco} = 1.72578$ and $\left|G(w_{pco})\right| = 2.27978$.
Following Lepschy et. al., 1986, two irrational reduced models have been proposed for this system and they are

- Reduced-model 1: $G_r(s) = \dfrac{k_1 e^{-\tau \cdot s}}{1 + d \cdot s + k_2 e^{-\tau \cdot s}}$
- Reduced-model 2 $G_r(s) = \dfrac{k e^{-\tau_1 \cdot s}}{s + k e^{-\tau_2 \cdot s}}$

Because the reduced models have delays in their denominators; the first step in the model reduction is the removal of the delays by approximation with a Pade approximant after which the system can be treated as before. It is noted that the delay in the numerator is not approximated. Note further that in Taiwo et. al. (2001) a rearrangement was used in order to handle the denominator delays exactly. The parameters of the optimal reduced model obtained for each of reduced-model 1 and 2 are as presented in Tables 3 and 4 respectively. An optimal second-order plus time delay reduced-model (using unit step input and h(t)=1 with ISE=0.018276) obtained for this system is:

$$G_r(s) = \frac{(2.78444 + 0.36683s)e^{-0.983628s}}{s^2 + 0.676287s + 2.78444}$$

Fig. 3 shows the open-loop step responses of the original and reduced models for unit step input and h(t)=1.
It is noted here that reduced models were also obtained for cases where inputs were a combination of step and impulse with similar results. These as well as the frequency responses have not been presented here because of space limitation.

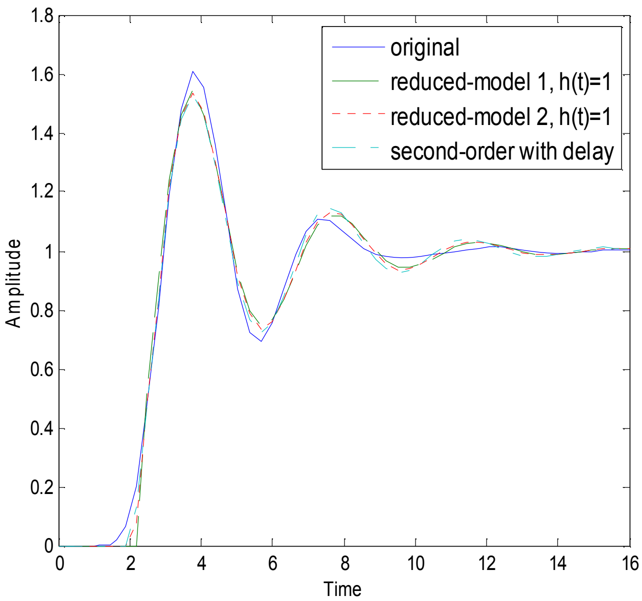

Fig. 3 Open-loop step responses of the original and reduced models for example 2

Table 3. Comparison of Optimal Model Parameters for Reduced-model 1

S/N	Method	k_1	k_2
1	This work, h(t)=1	2.180976	1.180875
2	This work, h(t)=t	2.232841	1.232848
3	This work, h(t)=t^2	2.220000	1.227800
4	Moment matching	4.1924	3.1924

Table 3. Comparison of Optimal Model Parameters for Reduced-model 1 (cont'd)

S/N	d	τ	ω_{pco}	$\left|G(\omega_{pco})\right|$	ISE
1	1.0912	1.2102	1.7055	2.34081	0.02837
2	1.1382	1.2037	1.7010	2.35966	0.02846
3	1.1340	1.2000	1.7065	2.33575	0.04074
4	3.1669	0.4368	3.7865	0.474213	0.05524

Table 4. Comparison of Optimal Model Parameters for Reduced-model 2

S/N	Method	k	τ_1
1	This work, $h(t) = 1$	1.243494	1.127150
2	Moment Matching	1.09535	0.981666
3	Taiwo, Effanga and Odunsanya	1.24488	1.128126

Table 4. Comparison of Optimal Model Parameters for Reduced-model 2 (cont'd)

S/N	τ_2	ω_{pco}	$\left|G(\omega_{pco})\right|$	ISE
1	0.837526	1.70673	2.45323	0.019898
2	0.894615	1.69706	1.80762	0.0490344
3	0.836682	1.70748	2.45611	0.0199004

3.3 Example 3

This is a two input two output model (Taiwo and Krebs, 1995) of an air compressor. The rational transfer function is

$$G(s) = \begin{bmatrix} \dfrac{0.1133e^{-0.715s}}{1.1783s^2 + 4.48s + 1} & \dfrac{0.9222}{2.071s + 1} \\ \dfrac{0.3378e^{-0.299s}}{0.361s^2 + 1.09s - 1} & \dfrac{-0.321e^{-0.94s}}{0.104s^2 + 2.463s + 1} \end{bmatrix}$$

To reduce it to a second order rational system, we assume that the reduced model is in the controllable form:
$\dot{x} = Ax(t) + Bu(t), \; y(t) = Cx(t)$
where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore

$$G_r(s) = C(sI - A)^{-1} B$$

As explained earlier, the computation time can be greatly reduced by obtaining the integrals element by element rather than lumping them together. For each element, the integral is then broken around the point where it has its delay. The parameters of the optimal reduced-order model for unit step input and different weighting functions are as presented in Table 5. Fig. 4 shows the open loop step responses of the original and reduced models for the (1, 1) element.

Similar agreements between reduced and original models were obtained for the other elements. The reduced models obtained when the inputs are a combination of step and impulse as well as frequency responses are omitted because of space constraints.

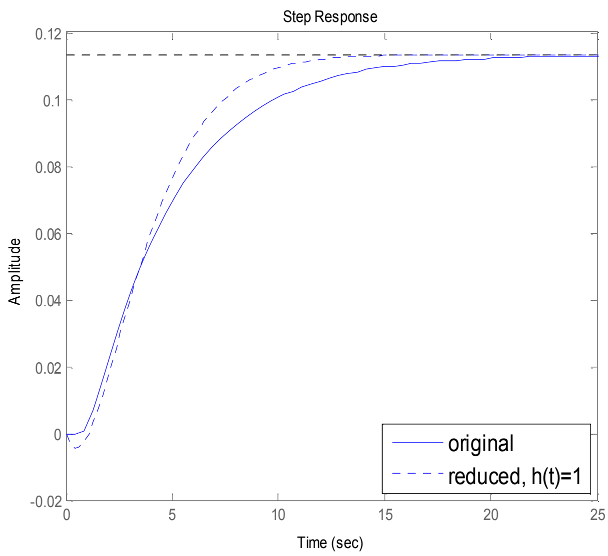

Fig. 4 Open-loop step responses of the original and reduced model for element (1,1)

3.4 Example 4

The original transfer function (Jerome and Ray, 1991) is

$$G(s) = \frac{(4s+2)e^{-8s} - (s+1)e^{-6s}}{(s+1)(2s+1)}$$

The model has $\omega_{pco} = 0.320026$ and $|G(\omega_{pco})| = 1.19554$

A first order plus time delay reduced model of the form

$$G_r(s) = \frac{c_0 e^{-\tau \cdot s}}{d_0 + s}$$

was obtained for the model. For this system ,the integral in Eq. (1) is broken at time 6, 8 and τ . The parameters of the optimal reduced models obtained, for unit step input and different weighting functions, are presented in Table 6. An optimal second-order plus time delay reduced-model (using unit step input and h(t)=1 with ISE=0.06150703) obtained for this system is:

$$G_r(s) = \frac{(1.16677 - 2.141s)e^{-6.65529s}}{s^2 + 1.6167s + 1.16677}$$

Fig. 5 shows the open-loop step responses of the original and reduced models. Results obtained for a combination of inputs as well as frequency responses could not be included because of space limitation.

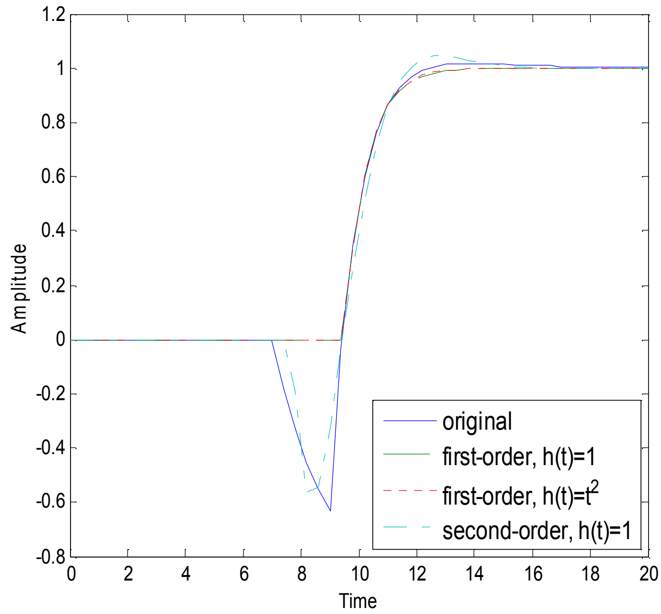

Fig. 5 Open loop responses of original and reduced models

Table 5. Optimal Reduced-Model Parameters for the Compressor Model

S/N	Method	A
1	This work, $h(t)=1$	$\begin{pmatrix} -0.564399 & -0.299517 \\ 0.0903509 & -0.412625 \end{pmatrix}$
2	This work, $h(t)=t$	$\begin{pmatrix} -0.817774 & 0.410386 \\ 0.104308 & -0.403535 \end{pmatrix}$
3	This work, $h(t)=t^2$	$\begin{pmatrix} -0.817774 & 0.410386 \\ 0.104308 & -0.403535 \end{pmatrix}$
4	Taiwo, Effanga and Odusanya [6], h(t)=1	$\begin{pmatrix} -0.5644 & 0.060790 \\ 0.09037 & -0.4126 \end{pmatrix}$

Table 5. Optimal Reduced-Model Parameters for the Compressor Model (cont'd)

S/N	C	ISE
1	$\begin{pmatrix} -0.019376 & 0.414458 \\ 0.219657 & -0.031275 \end{pmatrix}$	0.0086545
2	$\begin{pmatrix} -0.0035387 & 0.410386 \\ 0.309727 & -0.0155047 \end{pmatrix}$	0.0171484
3	$\begin{pmatrix} -0.0035387 & 0.410386 \\ 0.309727 & -0.0155047 \end{pmatrix}$	0.0548513
4	$\begin{pmatrix} -0.01939 & 0.4145 \\ 0.21965 & -0.03123 \end{pmatrix}$	0.008711

Table 6. First Order Reduced-Model Parameters for $\frac{(4s+2)e^{-8s}-(s+1)e^{-6s}}{(s+1)(2s+1)}$

S/N	Method	u_0	u_1	c_0	d_0
1	$h(t)=1$	0	1	1.30467	1.30467
2	$h(t)=t$	0	1	1.31606	1.31606
3	$h(t)=t^2$	0	1	1.33048	1.33048

Table 6. First Order Reduced-Model Parameters for $\frac{(4s+2)e^{-8s}-(s+1)e^{-6s}}{(s+1)(2s+1)}$ **(cont'd)**

S/N	τ	ω_{pco}	$\lvert G(\omega_{pco})\rvert$	ISE
1	8.46564	0.340908	0.967516	0.389476
2	8.46941	0.340999	0.968033	0.389484
3	8.47373	0.341126	0.968668	0.389514

4. CONCLUSIONS

The proposed algorithm has been applied to SISO and MIMO systems, rational, irrational retarded systems and the results obtained are found to agree with optimal reduced models available in the literature. With the algorithm, reduced order model can be easily made to have certain desired characteristics such as similar cross-over frequency and equal amplitude ratio at cross-over frequency. The ease with which results are obtained is also worth noting.

One of the uses of the reduced model is that it facilitates the parameterization of simple feedback controllers for the actual process using internal model control.

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