ROBUST MULTIVARIABLE CONTROL SYSTEM DESIGN USING THE METHOD OF INEQUALITIES

O. Taiwo[*], K. Owa[**], A. Akere[*], T. Ajetunmobi[*]

[*]*Chemical Engineering Department, Obafemi Awolowo University, Ile-Ife, Nigeria.*
(e-mail: femtaiwo@yahoo.com, ajebaba2001@yahoo.com)

[**]*Computer Science and Engineering Department, Obafemi Awolowo University, Ile-Ife, Nigeria.*

Abstract: This work is concerned with the design of robust control systems for multivariable time-delayed plants using the Method of Inequalities (MOI). It is an extension of previous applications in that the time delays are not approximated by rational functions. A further extension is that the designed systems were required to satisfy certain robustness conditions. This assures guaranteed stability and performance for uncertain systems.

Admirable qualities of the method are that it facilitates the design of simple controllers of predetermined structures, such as decentralized controllers, while allowing multi-objective specifications.

1.0 INTRODUCTION

The Method Of Inequalities (Zakian, 2005; Zakian, 1996; Taiwo, 1980; Zakian, and Al-Naib, 1973) uses numerical algorithms to solve the set of inequalities which characterize desired system performance and stipulated constraints.

The Simulink software utilized in this work facilitates exact representation of time delays in the design. The boundedness of the closed loop systems is guaranteed by checking the abscissa of stability of the transcendental characteristic equations and ensuring that it is negative. Satisfactory results have been obtained even for cases where initial controller parameters give rise to unstable closed-loop systems. Such systems are first stabilized using the moving boundaries process (MBP) available in MOI. System stability is thus always guaranteed using this method. Some case studies have been presented to show the applicability of this method to the design of robust control systems involving diagonal (decentralized) and full proportional plus integral controllers. The work is an exposition of the versatility of MOI in designing effective decentralized controllers for both nominal and uncertain systems.

2.0 PROBLEM FORMULATION

Figure 1 shows the basic control structure for feedback control. The controller and plant transfer functions K(s,p) and G(s) respectively are, in general, matrices of appropriate dimensions. The signals r(s), d(s) and y(s) respectively represent the input, disturbance and output vectors.
The objective is to compute the controller satisfying specified performance specifications and stipulated constraints.

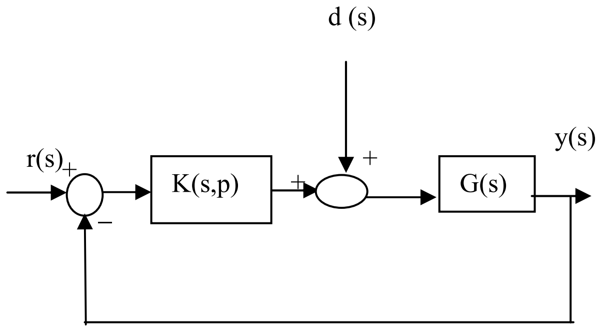

Fig. 1. Basic Control Structure Of Feedback Control

For two-input, two-output plant, the decentralized PI controller takes the form

$$K(s,p) = \begin{pmatrix} \frac{p_1 s + p_2}{s} & 0 \\ 0 & \frac{p_3 s + p_4}{s} \end{pmatrix} \quad (1)$$

The vector of performance functionals and constraints used is:

$$\Phi = (ov_1, rs_1, s_1, ov_2, rs_2, s_2, int_{21}, int_{12}, \mu_{rs}, \mu_{rp}, \varepsilon) \quad (2)$$

where ov, rs, s denote the overshoot, rise time and settling time respectively. The subscript represents the loop; int_{ab} ($a \neq b$) denotes the interaction for the unit step response of y_a

to r_b. The parameters ε, μ_{rs}, μ_{rp} denote abscissa of stability, maximum structured singular values (μ) for robust stability and robust performance respectively.

The full PI controller for a two-input, two-output plant is given by:

$$K(s,p) = \begin{pmatrix} \frac{p_1 s + p_2}{s} & \frac{p_3 s + p_4}{s} \\ \frac{p_5 s + p_6}{s} & \frac{p_7 s + p_8}{s} \end{pmatrix} \quad (3)$$

3.0 SOLUTION METHOD

Having specified the controller type, the controller design is started by entering an initial controller vector p°. Assuming the system is closed-loop stable, the performance functionals and constraints are easily evaluated from the simulation runs in the Simulink environment using the appropriate algorithms in MOI.

3.1 Testing for Closed-loop Stability

Recall that for closed-loop stability, all the roots of the transcendental characteristic equation

$$|I + GK| = 0 \quad (4)$$

must lie in the open left-half plane where I is an $n \times n$ identity matrix depicting the dimension of the system.

If the maximum value of the real part of the roots above is greater than the stability margin, the Moving Boundaries Process (MBP) (Zakian, 2005) is used to stabilize the system.

For a dynamical system, the desired performance and stipulated constraints can be expressed in terms of a conjunction of inequalities

$$\Phi_i(p) \le c_i \qquad i = 1,2,\ldots m \quad (5)$$

where p is a vector of controller parameters and c_i is a real number representing numerical bounds on particular aspect of dynamical behavior represented by $\Phi_i(p)$. In order to solve (5), if p^k denotes the kth successful trial value of p, then a suitable technique such as the Rosenbrock (1960) search algorithm is used to generate a new trial point $p^{\sim k}$ and with the aid of Simulink and Matlab routines in MOI, the vector of performance functionals $\Phi_i(p^{\sim k})$ is evaluated. If $p^{\sim k}$ is such that

$$\Phi_i(p^{\sim k}) < c_i^{\,k} \qquad i = 1,2,\ldots m \quad (6)$$

where

$$c_i^k = \begin{cases} c_i & \text{if } \Phi_i(p^k) < c_i \\ \Phi_i(p^k) & \text{if } \Phi_i(p^k) > c_i \end{cases} \quad (7)$$

then $p^{\sim k}$ is accepted and becomes the new point p^{k+1}. If (6) does not hold, another trial point is investigated until success defined by (6) occurs. The process continues until the inequalities are solved or the maximum number of iterations has been reached.

To guarantee stability at every iteration, we introduce the inequality:

$$\Phi_{m+1}(p) \le \varepsilon, \quad (8)$$

where the abscissa of stability, $\varepsilon < 0$. For computing purposes, we set ε to a value of -0.01 in this work.
Furthermore, $\Phi_{m+1}(p)$ is evaluated using the equation:

$$\Phi_{m+1}(p) = \max \{\mathrm{Re}[\lambda_i(p)]\} \quad (9)$$

Where $\lambda_i(p)$ are the roots of (4).

3.2 Design for Robustness

To design for robustness, we redraw the MIMO system as a block diagram with input uncertainty and with external inputs as shown in Fig. 2.
The weighting functions w_I and w_p are chosen to characterize the effect of uncertainties in the plant input and output respectively (Gu, *et al*, 2005). The input and output to the system are r and θ respectively. We assume that there is a multiplicative input uncertainty to the plant as shown in Fig. 2. The input weight is given as $|w_I|$ I where, following Taiwo and Krebs (1996)

$$w_I = 0.1(5s+1)/(0.4s+1) \quad (10)$$

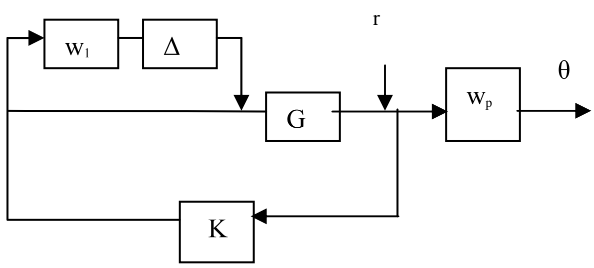

Fig. 2. Block Diagram of plant with input uncertainty and with external inputs

This means that the error in the input is more than 100% at high frequencies and up to 10% at steady state. The performance weight is given by $w_p I$ where

$$w_p = 0.34(15s+1)/15s \quad (11)$$

The integral action in (11) ensures no steady state error for step inputs and the factor 0.34 indicates a maximum high frequency error amplification of just less than 3. We now reduce the model in Fig. 2 to a Linear Fractional Transformation (LFT) model in Fig. 3 so that Doyle's structured singular value (μ) analysis (Taiwo and Krebs, 1996) could be applied. The matrices, Δ and M in the LFT structure of Fig. 3 are easily derived:

$$\mathrm{M} = \begin{bmatrix} -w_1 KSG & w_1 KS \\ -w_p SG & w_p S \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (12)$$

where S is the sensitivity function:

$$\mathrm{S} = (\mathrm{I} + \mathrm{GK})^{-1} \quad (13)$$

And Δ takes the form:

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \quad (14)$$

Since the actions of the actuators are usually independent of each other, it is reasonable to model the perturbation Δ_1 as a diagonal 2×2 block while Δ_2 ,which depicts a fictitious perturbation, is a full 2×2 block. If μ_{rs} and μ_{rp} are the maximum structured singular values of M_{11} and M respectively, then the conditions for robust stability and robust performance are such that:

$$\text{Robust stability: } \mu_{rs} < 1 \quad (15)$$
$$\text{Robust performance: } \mu_{rp} < 1 \quad (16)$$

The μ-analysis and synthesis toolbox in MATLAB was used for computing the structured singular values for robust stability and performance. These are added as constraints in the determination of controller parameters using the Method of Inequalities.

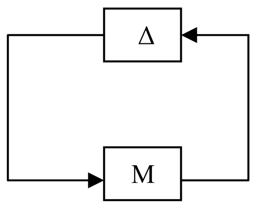

Fig. 3. Linear Fractional Transformation, Δ-M structure for studying effect of uncertainty on stability or performance.

4.0 ILLUSTRATIVE EXAMPLES

Eight case studies of robust control of MIMO systems with time delays are hereby given using the Method of Inequalities.

Example 1: The plant transfer function for the methanol-water distillation column modeled by Wood and Berry (1973) is:

$$G(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \quad (17)$$

For the decentralized PI controller, the performance bounds are:
c_1= 0.2, c_2= 15, c_3= 32, c_4= 0.2, c_5= 15, c_6= 32, c_7= int_{21}=0.5, c_8= int_{12}=0.5, c_9=1, c_{10}= 1, ε = -0.01, where c_1 and c_4 denote the overshoot in loop 1 and loop 2, c_3 and c_6 denote the settling time in loop 1 and loop 2, c_2 and c_5 denote the rise-time in loop 1 and loop 2, int_{21} denotes the interaction for the unit step response of y_2 to r_1, ε denotes the stability margin, c_9 and c_{10} denote bounds on μ values for robust stability and performance respectively. The initial controller parameters are p°= [0.1644 0.0179 -0.0581 -0.0093]. After 49 iterations, the final controller parameters are p = [0.2219 0.0211 -0.0748 -0.0120]. The vector of performance functionals is Φ = [0 9.2450 33.0943 0.0339 11.2162 31.9481 0.4891 0.3614 0.2596 0.9932 -0.0369]. It is clear that the system has both stability and performance robustness. The closed-loop system responses using this diagonal PI controller are shown in Fig.4. In all the figures, σ_{ij} denotes the unit step response of y_i to r_j.

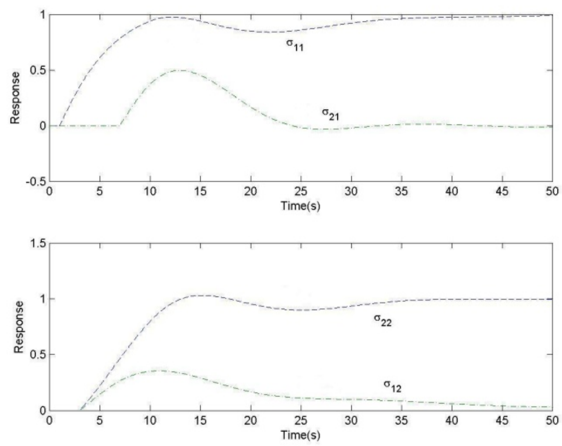

Fig. 4. Closed-Loop Response using Decentralized Controller for the Wood and Berry Plant.

For the full PI controller, the performance bounds are :
c_1= 0.1, c_2= 10, c_3= 25, c_4= 0.1, c_5= 10, c_6= 25, c_7=int_{21} =0.55,c_8= int_{12}=0.55, c_9=1, c_{10}= 1, ε = -0.01. The initial controller parameters are p°= [0.1778 0.0194 0 0 0 0 -0.0959 -0.0153]. After 41 iterations, the final controller parameters are p = [0.1789 0.0194 0 -0.005 -0.02 0 -0.0959 -0.0153]. The vector of performance functionals is Φ = [0.0045 9.9881 28.8233 0.1015 9.4125 27.8633 0.4503 0.3849 0.2997 1.0698 -0.0378]. It is clear that the system has stability robustness and marginally exceeds the performance robustness

constraint. Fig. 5 shows the responses obtained with this full PI controller.

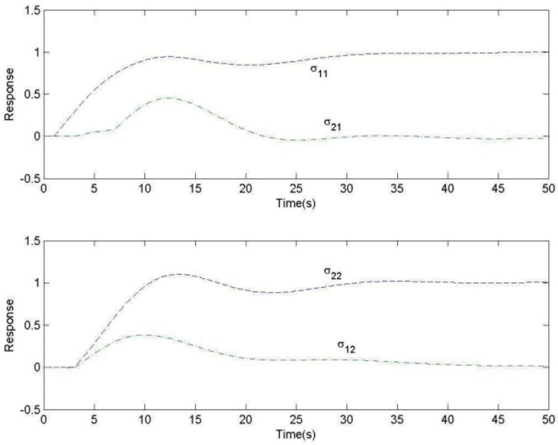

Fig. 5. Closed-Loop Response using Full Controller for the Wood and Berry Plant

Example 2: The plant transfer function for the plant modeled by Nordfeldt and Hagglund (2006) is :

$$G(s) = \begin{bmatrix} \dfrac{3e^{-3s}}{9s+1} & \dfrac{2e^{-2s}}{6s+1} \\ \dfrac{1e^{-4s}}{5s+1} & \dfrac{2e^{-4s}}{7s+1} \end{bmatrix} \quad (18)$$

A similar procedure as for example 1 was used here. For the decentralized PI controller, the performance bounds are: c_1= 0.2, c_2= 3, c_3= 12, c_4= 0.2, c_5= 3, c_6= 12, c_7= int_{21}=0.5, c_8= int_{12}=0.5, c_9=1, c_{10}= 1, ε= -0.01. The initial controller parameters are p°= [0.5465 0.0792 0.205 0.0457]. After 34 iterations, the final controller parameters are p = [0.5627 0.0816 0.205 0.0457]. With μ_{rs}=0.2796 and μ_{rp} =0.9971, it is clear that the system has both stability and performance robustness. The closed-loop system responses using this decentralized PI controller are shown in Fig. 6.

For the full PI controller, the performance bounds are: c_1= 0.2, c_2= 10, c_3= 25, c_4= 0.2, c_5= 10, c_6= 25, c_7= int_{21}=0.5, c_8=int_{12}=0.5, c_9=1, c_{10}=1, ε= -0.01. Note that some of the bounds have been increased here since they were not satisfied by the decentralized controller. The initial controller parameters are p°= [0.5781 0.0816 0 0 0 0 0.2050 0.0457]. After 75 iterations, the final controller parameters are p= [0.6169 0.0816 -0.0163 -0.0022 -0.0500 -0.0059 0.1967 0.0462]. With μ_{rs}=0.2746 and μ_{rp} =0.9844, it is also clear that the system has both stability and performance robustness. Fig. 7 shows the responses of the closed-loop system using this full PI controller.

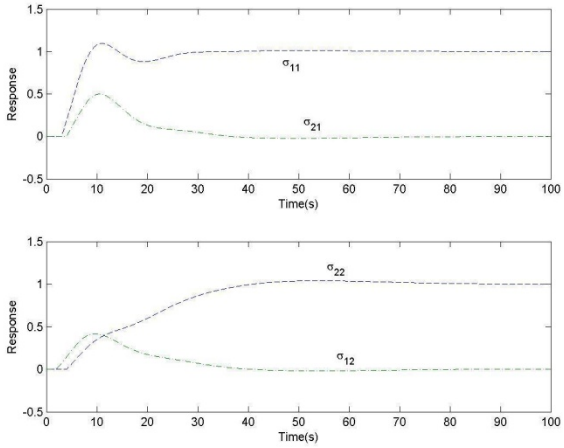

Fig. 6. Closed-Loop Response using Decentralized Controller for the Norfeldt and Hagglund Plant.

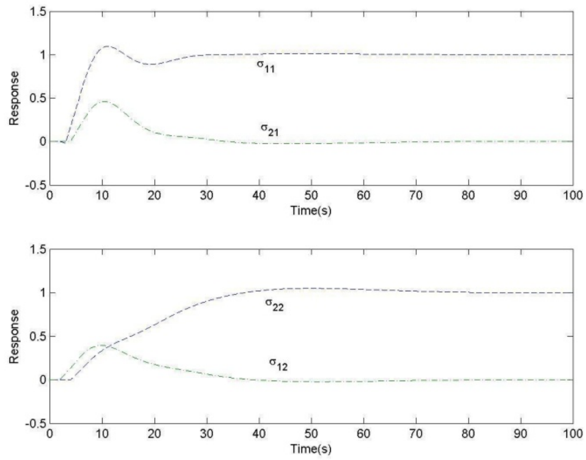

Fig. 7. Closed-loop Response of Nordfeldt and Hagglund Plant using Full Controllers

Example 3: This example demonstrates the scenario where the initial parameter vector results in a closed-loop unstable system. The plant transfer function (Luyben and Vinante, 1972) is:

$$G(s) = \begin{bmatrix} \dfrac{-2.2e^{-s}}{7s+1} & \dfrac{1.3e^{-0.3s}}{7s+1} \\ \dfrac{-2.8e^{-1.8s}}{9.5s+1} & \dfrac{4.3e^{-0.35s}}{9.2s+1} \end{bmatrix} \quad (19)$$

For the decentralized PI controller, the performance bounds are: c_1= 0.2, c_2= 3, c_3= 12, c_4= 0.2, c_5= 3, c_6= 12, c_7= int_{21}=0.5, c_8= int_{12}=0.5, c_9=1, c_{10}= 1, ε= -0.01. The initial parameter vector, p° = [-1 -1 2 -2] resulted in a closed-loop unstable system. MBP was used to stabilize the system, yielding p= [-1.2 -1.2 4.6 0.60] after 12 iterations. MOI could now proceed to solve the inequalities. After 43 iterations, the final controller parameters being p = [-1.2 -1.2 3.68 0.6].

With μ_{rs}=1.03 and μ_{rp} = 1.61, the system marginally exceeds robust stability constraint and does not satisfy the performance robustness constraint as well.
Fig. 8 shows the closed-loop responses using this decentralized controller.

For the full PI controller, the performance bounds are:
c_1= 0.2, c_2= 3, c_3= 10, c_4= 0.2, c_5= 3, c_6= 10, c_7= int_{21}=0.5, c_8= int_{12}=0.5, c_9=1, c_{10}= 1, ε= -0.01. The initial controller parameters are p°= [-1.7 -0.8 0.4 -0.005 -0.065 -0.02 2.9 0.8]. After 45 iterations, the final controller parameters are
p = [-1.7192 -0.7993 0.400 -0.0053 0.0195 -0.02 2.9237 0.8359]. With μ_{rs}=0.9445 and μ_{rp} =1.6399, the full controller satisfies robust stability constraint but does not satisfy the performance robustness constraint. With a PI controller, it was not possible to design a system satisfying the performance robustness constraint although it should be possible to satisfy the constraint with more complicated controllers. Fig. 9 shows the closed-loop responses using this full PI controller.

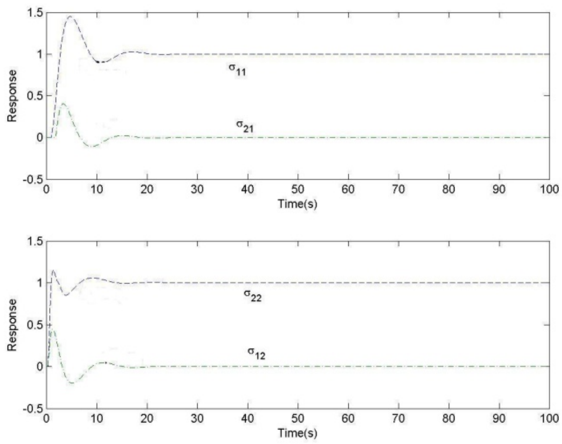

Fig. 8. Closed-Loop Response using Decentralized Controller for the Luyben and Vinante Plant.

Example 4: The plant transfer function for the 3-input,3-output Tyreus Case-4 plant (1979) is:

$$G(s) = \begin{bmatrix} \dfrac{-1.986e^{-0.71s}}{66.67s+1} & \dfrac{5.24e^{-60s}}{400s+1} & \dfrac{5.984e^{-2.24s}}{14.29s+1} \\ \dfrac{0.0204e^{-0.59s}}{(7.14s+1)^2} & \dfrac{-0.33e^{-0.68s}}{(2.38s+1)^2} & \dfrac{2.38e^{-0.42s}}{(1.43s+1)^2} \\ \dfrac{0.374e^{-7.7s}}{22.22s+1} & \dfrac{-11.3e^{-3.79s}}{(21.74s+1)^2} & \dfrac{-9.811e^{-1.59s}}{11.36s+1} \end{bmatrix} \quad (20)$$

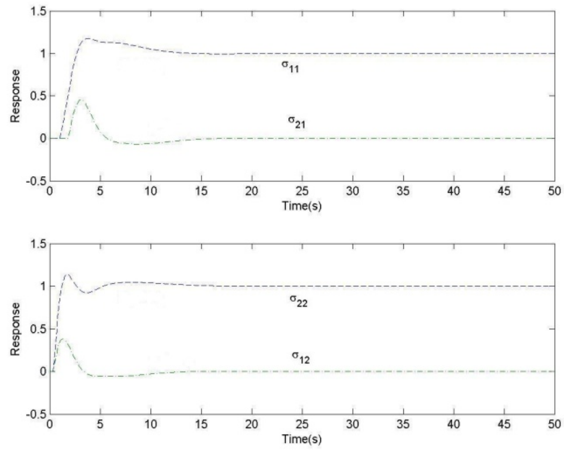

Fig. 9. Closed-Loop Response using Full Controller for the Luyben and Vinante Plant.

The vector of performance functionals is :

$$\Phi = (ov_1, rs_1, s_1, ov_2, rs_2, s_2, ov_3, rs_3, s_3, int_{21}, int_{31}, int_{12}, int_{32}, int_{13}, int_{23}, \mu_{RS}, \mu_{RP}, \varepsilon) \quad (21)$$

where the parameters ov, rs, s, ε, μ_{RS}, μ_{RP} , int_{ab} have been defined in (2).

For the decentralized PI controller, the performance bounds are:
c_1= 0.2, c_2= 30, c_3= 100, c_4= 0.2, c_5= 30, c_6= 100, c_7= 0.2, c_8= 30, c_9= 100, c_{10}= int_{21}=0.5, c_{11}= int_{31}=0.5, c_{12}= int_{12}=0.5, c_{13}= int_{32}=0.5, c_{14}= int_{13}=0.5, c_{15}= int_{23}=0.5, c_{16}=1, c_{17}= 1, ε= -0.01. The initial controller parameters are p°= [-14 -1 -1.8 -0.06 -0.3 -0.005]. After 15 iterations, the final controller parameters are p = [-14.8899 -1.0859 -1.8026 -0.0636 -0.3566 -0.0044]. With μ_{rs}=0.4036 and μ_{rp} =1.2484, the diagonal PI controller satisfies robust stability constraint but does not satisfy the performance robustness constraint. Fig.10 shows the responses of the closed-loop system using this decentralized controller.

For the full PI controller, the performance bounds are:
c_1= 0.1, c_2= 10, c_3= 15, c_4= 0.1, c_5= 10, c_6= 15, c_7= 0.1, c_8= 10, c_9= 15, c_{10}= int_{21}=0.4, c_{11}= int_{31}=0.4, c_{12}= int_{12}=0.4, c_{13}= int_{32}=0.4, c_{14}= int_{13}=0.4, c_{15}= int_{23}=0.4, c_{16}=1, c_{17}= 1, ε= -0.01. The initial controller parameters are p°= [-13.4 -0.44 0.03 0.4 0.03 0.4 -0.2 0.0006 -1.8 -0.03 -0.2 0 -0.065 -0.005 -0.005 0.04 -0.36 -0.005]. After 18 iterations, the final controller parameters are
p = [-13.4009 -0.4398 0.0325 0.4000 0.0325 0.400 -0.200 0.0006 -1.7913 -0.0344 -0.200 0 -0.065 -0.005 -0.005 0.04 -0.3611 -0.0045]. With μ_{rs}=0.4261 and μ_{rp} =1.0126, it is clear that the system has stability robustness and marginally exceeds the performance robustness constraint. Fig. 11 shows the closed-loop responses using this

full PI controller. The figure clearly demonstrates the effectiveness of the full PI controller in reducing σ_{32}.

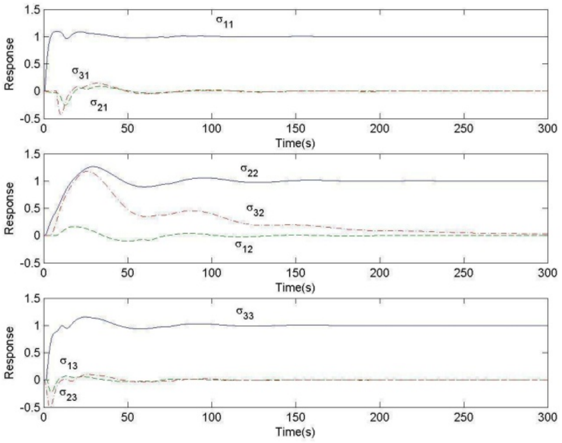

Fig. 10 Response of Tyreus Case 4 Plant to Unit Step set-point increase using Diagonal Controllers

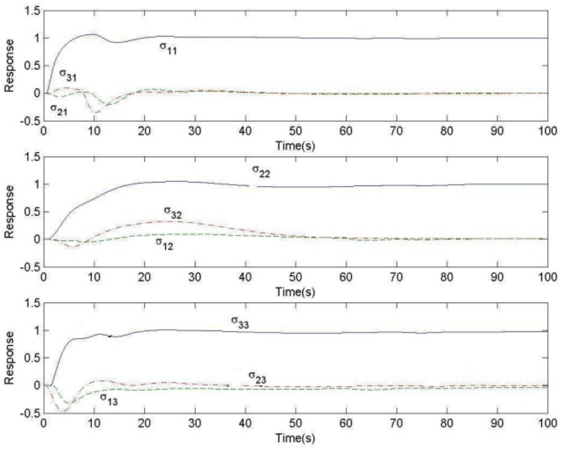

Fig. 11. Response of Tyreus Case 4 Plant to Unit Step set-point increase using Full Controllers

5. CONCLUSION

Robust controllers have been designed for MIMO systems using MOI. As expected, it is easier to search for controller parameters such that $\mu_{rs}<1$ than those for $\mu_{rp}<1$. This method facilitates performance in terms of multi-objective specifications unlike classical optimization which expresses performance as a single performance index. The automation of controller "tuning", which MOI does, confers rapid computation of acceptable controller parameters.

In general, a full PI controller gives a system with small interaction, less overshoot, smaller settling time and better robustness than the decentralized controllers. However decentralized PI controllers can be easily implemented using two conventional analog PI controllers and have generally given rise to systems with acceptable performance.

REFERENCES

Gu, D.W, Hr. Petkov and M.M Konstantinov (2005). *Robust Control Design with Matlab.* Springer-Verlag ,London.

Luyben, W.L. and Vinante, C.D (1972). Experimental studies of distillation decoupling, *Kem, Teolliscus*. 499-514.

Nordfeldt, P. and T. Hagglund (2006). Decoupler and PID controller design of TITO systems. *Journal of Process Control,* **16**, 923-936.

Rosenbrock, H.H (1960). An automatic method for finding the greatest and least value of a function. *Comp. J.,***3** 175-184.

Taiwo, O. and Krebs, V. (1996). Robust Control System Design for Plants with Recycle. *The Chemical Engineering Journal,* **62,** 1-6.

Taiwo, O. (1980). Application of the Method Of Inequalities to the Multivariable control of binary distillation columns. *Chem.Eng.Sci,* **32,** 847-858..

Tyreus, B.D (1979). Multivariable control system design for an industrial distillation column. *Ind.Eng.Chem.Proc.Des.Dev.,* **18,** 177-182.

Wood, R.K. and Berry, M.W (1973). Terminal composition control of a binary distillation column. *Chem.Eng. Sci.,* **28,** 1707-1717.

Zakian, V. (2005). Foundations of Control Systems Design , In: *Control Systems Design, A New Framework,* (Zakian, V. (Ed)), 3-68, Springer-Verlag, London.

Zakian, V. (1996). Perspectives on the Principles of Matching and the Method of Inequalities, *Int.J.Contr.*, **65,** 147-175.

Zakian,V. and U. Al-Naib,(1973) "Design of Dynamical and control systems by the method of Inequalities, Proc.IEE, **120**, 1421-1427.