Sliding Mode Controllers Using Output Information: An LMI Approach

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Abstract: This paper considers the development of Sliding mode output feedback controllers. The existence problem is solved via a static output feedback formulation using a descriptor approach. Linear matrix inequalities (LMI) are used to obtain the parameters of the switching function. The paper provides conditions in terms of the system structure for a stable reduced-order sliding motion to exist. A controller is constructed to ensure the sliding mode is reached. A numerical example from the literature illustrates the proposed method.

Keywords: sliding mode control, static output feedback, descriptor representation, stabilization, linear matrix inequalities (LMI)

1. INTRODUCTION

Sliding Mode Control SMC is known for its complete robustness to so-called matched uncertainties and disturbances (Utkin 1992). Many early theoretical developments in SMC assume that all the system states are accessible. This is however very limiting when considering applications of practical relevance in industry, where only a subset of the states maybe available in synthesizing a control law. In this case, either an observer can be used to estimate the unmeasured states or a tractable framework for static output feedback sliding mode control (SOFSMC) should be constructed. The latter problem is widely studied as it has less computational and hardware costs compared to observer-based approaches.

There are two difficulties in the design of SOFSMC. One is the existence problem, i.e., the design of a switching surface in the output vector space which is usually of lower order than the state vector space. Zak and Hui (1993) and El-Khazahi and Decarlo (1995) proposed two different methods to design sliding surfaces using eigenvalue assignment and eigenvector techniques. Edwards and Spurgeon (1995) provided a canonical form via which the SOFSMC design problem is converted to a static output feedback stabilization SOFS problem. As stated by Edward *et al.*(2003), all previous-reported methods for the existence problem are, in fact, equivalent to a particular SOF problem. The solution to the general SOF problem, even for linear time-invariant systems, is still open.

Linear matrix inequality methods have been considered within the context of sliding mode controller design. For example, Edwards *et al.* (2000, 2001) presented LMI methods to design static sliding mode output feedback controllers and Choi (2002) presented conditions to solve the existence problem in terms of LMIs for linear uncertain systems. The other difficulty is the synthesis problem. It is non-trivial to synthesize a control law only using the output vector, since the derivative of the sliding surface is always related to the unmeasured states. In this paper a descriptor approach is applied to derive LMIs for the solution of the sliding mode control output feedback problem. An example from the literature illustrates the efficiency of the method. In section 2 the problem formulation is described and the general framework is described in section 3. A solution to the existence problem is presented in section 4. Section 5 shows the formulation for the reachability problem and an example is demonstrated in Section 5.

The notation used in the paper is quite standard. The symbol $\mathcal{R}$ is used to represent a set of real numbers and $\|\cdot\|$ denotes the Euclidean norm of a vector and the induced spectral norm of a matrix.

2. PROBLEM FORMULATION

Consider an uncertain dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t,x,u), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $y \in \mathcal{R}^p$ with $m \le p \le n$. Assume that the nominal linear system (A, B, C) is known and that the input and output matrices B and C are both of full rank. The unknown function f: $\mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, which represents the system non-linearities plus any model uncertainties in the system, is assumed to satisfy the matching condition

$$f(t,x,u) = B\xi(t,x,u) \tag{2}$$

where the bounded function $\xi : \mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^m$ satisfies

$$\|\xi(t,x,u)\| < k_1\|u\| + a(t,y) \tag{3}$$

for some known function $a : \mathcal{R}_+ \times \mathcal{R}^p \to \mathcal{R}_+$ and positive constant $k_1 < 1$.

Initially the intention will be to explore when static output feedback sliding mode control can be employed. A control law will be sought which induces an ideal sliding motion on the surface

$$\mathcal{S} = \{x \in \mathcal{R}^n : FCx = 0\} \tag{4}$$

for some selected matrix $F \in \mathcal{R}^{m\times p}$ of the form

$$u(t) = -\gamma F y(t) - v_y \tag{5}$$

where γ is a design parameter and the discontinuous vector

$$v_y(t) = \begin{cases} -\rho(t,y)\dfrac{Fy(t)}{||Fy(t)||} & \text{if } \; Fy \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where $\rho(t,y)$ is some positive scalar function of the outputs

$$\rho(t,y) = (k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2)/(1-k_1)$$

where γ and γ_2 are positive design scalars.

3. A GENERAL FRAMEWORK

Consider the system in (1) and assume rank $(CB) = m$. The reason for imposing this rank restriction at the outset is that for a unique equivalent control to exist, the matrix $FCB \in \mathcal{R}^{m\times m}$ must have full rank. It is well known that

$$rank(FCB) \leq min\{rank(F), rank(CB)\} \tag{7}$$

and so in order for FCB to have full rank both F and CB must have rank m. The matrix F is a design parameter and therefore by choice can be chosen to be of full rank. A necessary condition therefore for the matrix FCB to be full rank is that $rank(CB) = m$.

The first problem which must be considered is how to choose F so that the associated sliding motion is stable. A control law analogous to (5) and (6) will then be used to guarantee the existence of a sliding motion.

3.1 Hyperplane Design

It can be shown that if $rank(CB) = m$ there exists a coordinate system in which the triple (A, B, C) has the structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \; B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \; C = [\,0 \;\; T\,] \tag{8}$$

where $B_2 \in \mathcal{R}^{m\times m}$ is non-singular and $T \in \mathcal{R}^{p\times p}$ is orthogonal (Edwards, Spurgeon, 1995). Furthermore, $A_{11} \in \mathcal{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks in the system matrix are partitioned accordingly.

Let

$$[\,F_1 \;\; F_2\,] = FT \tag{9}$$

where $F_1 \in \mathcal{R}^{p-m}$ and $F_2 \in \mathcal{R}^m$. As a result

$$FC = [\,F_1C_1 \;\; F_2\,] \tag{10}$$

where

$$C_1 = \left[\,0_{(p-m)\times(n-p)} \;\; I_{(p-m)}\,\right] \tag{11}$$

Therefore $FCB = F_2B_2$ and the square matrix F_2 is nonsingular. By assumption the uncertainty is matched and therefore the sliding motion is independent of the uncertainty. In addition, because the canonical form in (8) can be viewed as a special case of the regular form normally used in sliding mode controller design, the reduced-order sliding motion is governed by a free motion with system matrix

$$A_{11}^s = A_{11} - A_{12}F_2^{-1}F_1C_1 \tag{12}$$

which must therefore be stable. If $K \in \mathcal{R}^{m\times(p-m)}$ is defined as $K = F_2^{-1}F_1$ then

$$A_{11}^s = A_{11} - A_{12}KC_1 \tag{13}$$

and the problem of hyperplane design is equivalent to a static output feedback problem for the system (A_{11}, A_{12}, C_1), where (A_{11}, A_{12}) is controllable and (A_{11}, C_1) is observable.

4. EXISTENCE PROBLEM

It will be shown that system (8) is stable if the reduced order system (13) is stable. Consider a Lyapunov function $V = x_1^T P x_1$ with

$$\dot{x}_1 = (A_{11} - A_{12}KC_1)x_1 \tag{14}$$

Using the descriptor method as in Fridman (2001) and the free-weighting matrices technique from He *et al* (2007), the right-hand side of the expression

$$0 = 2[x_1^T P_2^T + \dot{x}_1^T P_3^T][-\dot{x}_1 + (A_{11} - A_{12}KC_1)x_1]$$

with matrix parameters $P_2, P_3 = \epsilon P_2 \in \mathcal{R}^{n-m}$ is added into the right hand-side of $\dot{V} = 2\dot{x}_1^T P x_1$. It is necessary to find the conditions that guarantee that

$$\begin{aligned} \dot{V} &= 2\dot{x}_1^T P x_1 + 2[x_1^T P_2^T + \dot{x}_1^T P_3^T][-\dot{x}_1 \\ &+ (A_{11} - A_{12}KC_1)x_1] < 0 \end{aligned} \tag{15}$$

Setting $\eta = col\{x_1, \dot{x}_1\}$ it follows that

$$\dot{V} = \eta^T \Theta \eta \leq 0 \tag{16}$$

if the matrix inequality

$$\Theta = \begin{bmatrix} P_2^T(A_{11} - A_{12}KC_1) + (A_{11} - A_{12}KC_1)^T P_2 & P - P_2^T + \varepsilon(A_{11} - A_{12}KC_1)^T P_2 \\ * & -\varepsilon P_2 - \varepsilon P_2^T \end{bmatrix} < 0$$

is feasible. Multiplying the latter inequality from the right and the left by $diag\{P_2^{-1}, P_2^{-1}\}$ and its transpose respectively and denoting $Q_2 = P_2^{-1}$, $\bar{P} = Q_2^T P Q_2$, $\Theta < 0$ if and only if

$$\bar{\Theta} = \begin{bmatrix} (A_{11} - A_{12}KC_1)Q_2 + Q_2^T(A_{11} - A_{12}KC_1)^T & \bar{P} - Q_2 + \varepsilon Q_2^T(A_{11} - A_{12}KC_1)^T \\ * & -\varepsilon Q_2 - \varepsilon Q_2^T \end{bmatrix} < 0$$

Choose the LMI variable Q_2 in the following form

$$Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}M & \delta Q_{22} \end{bmatrix} \tag{17}$$

where Q_{22} is a $(p-m)\times(p-m)$-decision variable and M is some $(p-m)\times(n-p)$-matrix and δ is a tuning parameter. Then it can be verified that

$$KC_1Q_2 = [\, KQ_{22}M \quad \delta KQ_{22}\,]\,. \tag{18}$$

Defining

$$Y = KQ_{22} \tag{19}$$

It follows

$$KC_1Q_2 = [\, YM \quad \delta Y\,] \tag{20}$$

The matrix K can be found by solving the following LMI with the tuning parameters δ and ε:

$$\bar{\Theta} = \begin{bmatrix} \begin{array}{c} A_{11}Q_2 - A_{12}[YM \quad \delta Y] \\ +Q_2^TA_{11}^T - [YM \quad \delta Y]^TA_{12}^T \end{array} & * \\ \begin{array}{c} \bar{P} - Q_2 + \varepsilon Q_2^TA_{11}^T \\ -\varepsilon[YM \quad \delta Y]^TA_{12}^T \end{array} & -\varepsilon Q_2 - \varepsilon Q_2^T \end{bmatrix} < 0 \tag{21}$$

Theorem: Given scalars ϵ, δ and a matrix $M \in \mathcal{R}^{(p-m)\times(n-p)}$, if there exists a $(n-m)\times(n-m)$ symmetric matrix $\bar{P} > 0$, and matrices $Q_{22} \in \mathcal{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathcal{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathcal{R}^{(n-p)\times(p-m)}$, $Y \in \mathcal{R}^{m\times(p-m)}$ such that LMI (21) holds, then the reduced order system (13) is asymptotically stable.

Once K has been synthesized, choose

$$F = F_2\,[\, K \;\; I_m\,]\,T^T \tag{22}$$

The following section develops conditions to ensure that the uncertain system is quadratically stable and an ideal sliding motion is induced on $\mathcal{S}$ in finite time.

5. REACHABILITY PROBLEM

It can be shown that there exist a coordinate system in which the system triple $(\bar{A}, \bar{B}, F\bar{C})$ has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \; \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \; F\bar{C} = [\, 0 \;\; F_2\,] \tag{23}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$. Let $\bar{P}$ be a symmetric positive definite matrix partitioned conformably with the matrix in (23) so that

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \tag{24}$$

then the matrix $\bar{P}$ satisfies the structural constraint

$$\bar{P}\bar{B} = \bar{C}F^T \tag{25}$$

if the design matrix $F_2 = B_2^T\bar{P}_2$. The matrix $\bar{P}$ can be shown to be a Lyapunov matrix for $A_0 = \bar{A} - \gamma\bar{B}F\bar{C}$ for all $\gamma > \gamma_0$, where γ_0 is defined to ensure $L(\gamma) = \bar{P}A_0 + A_0^T\bar{P}$ is negative definite (Edwards *et al.* 1995).
In the new coordinate system the uncertain system (1) can be written as

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}(u(t) + \xi(t, z, u)) \tag{26}$$

Proposition: (Edwards and Spurgeon, 1995) The variable structure control law (5) quadratically stabilizes the uncertain system given in (26).

Proof: Consider as a candidate Lyapunov function the positive definite expression

$$V(z) = z^T\bar{P}z \tag{27}$$

Taking derivatives along the system trajectory and using the structural constraint from (25) gives

$$\begin{aligned} \dot{V} &= z^T(\bar{A}^T\bar{P} + \bar{P}\bar{A} - 2\gamma(F\bar{C})^TF\bar{C})z \\ &\quad +2z^T\bar{P}\bar{B}(\xi - v_y) \\ &= z^TL(\gamma)z + 2y^TF^T(\xi - v_y) \\ &\le z^TL(\gamma)z - 2y^TF^Tv_y + 2\|Fy\|\|\xi\| \\ &= z^TL(\gamma)z - 2\rho(t,y)\|Fy\| + 2\|Fy\|\|\xi\| \\ &< z^TL(\gamma)z - 2\|Fy\|(\rho(t,y) - k_1\|u\| - \alpha(t,y)) \end{aligned}$$

But by definition

$$\rho(t,y) = (k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2)/(1 - k_1)$$

and so by rearranging

$$\begin{aligned} \rho(t,y) &= k_1\rho(t,y) + k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2 \\ &\ge k_1(\|v_y\| + \gamma\|Fy\|) + \alpha(t,y) + \gamma_2 \\ &\ge k_1\|u\| + \alpha(t,y) + \gamma_2 \end{aligned} \tag{28}$$

Using (28) in the inequality for the Lyapunov derivative

$$\dot{V} < z^TL(\gamma)z - 2\gamma_2\|Fy\| < 0 \quad \text{if } z \neq 0 \text{ and } \gamma > \gamma_0$$

and therefore the system is quadratically stable.

Corollary: An ideal sliding motion takes place on the surface S in the domain

$$\Omega = \{z \in \mathcal{R}^n \;\;:\;\; \|B_2^{-1}A_0^L\|\|z\| < \gamma_2 - \eta\}$$

where matrix A_0^L represents the last m rows of A_0 and η is a small scalar satisfying $0 < \eta < \gamma_2$.

Proof
Substituting from equation (5) it follows from (26) and (4) that

$$\dot{s} = F\bar{C}A_0z + F_2B_2(\xi - v_y)$$

Let $V_c : \mathcal{R}^m \to \mathcal{R}$ be defined by

$$V_c(s) = 2s^T(F_2^{-1})^T\bar{P}_2F_2^{-1}s$$

Then using the fact that $F_2^T = \bar{P}_2B_2$ it follows that

$$(F_2^{-1})^T\bar{P}_2F_2^{-1}F\bar{C}A_0 = B_2^{-1}A_0^L$$

Then it can be verified that

$$\begin{aligned} \dot{V}_c &= 2s^TB_2^{-1}A_0^Lz + 2s^T(\xi - v_y) \\ &\le 2\|s\|\|B_2^{-1}A_0^Lz\| - 2\gamma_2\|s\| \\ &< -2\eta\|s\| \end{aligned} \tag{29}$$

if $z \in \Omega$. So there exists a t_0 such that $z(t) \in \Omega$ for all $t > t_0$. Consequently (29) holds for all $t > t_0$. A sliding motion will thus be attained in finite time. □

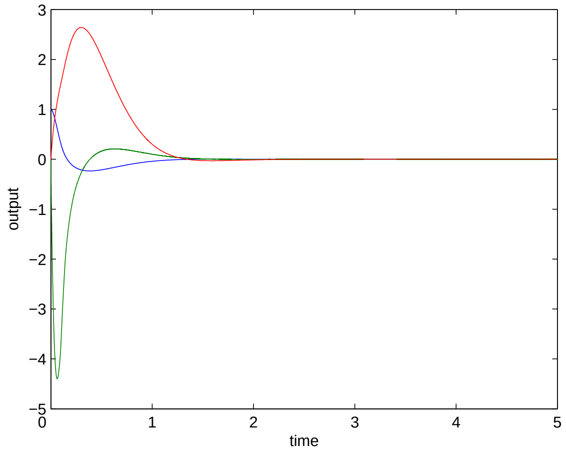

Fig. 1. **Output Against time**

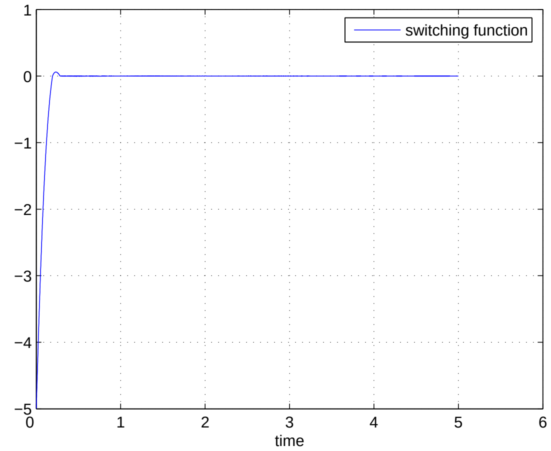

Fig. 2. **Switching function**

Evidence of the efficiency of the method will be demonstrated by considering an example taken from the literature. (Edwards. et. al 2003)

6. EXAMPLE

Consider the fourth order system

$$A = \begin{bmatrix} -2.724 & -13.808 & 0 & 0 \\ 0.73 & -4.782 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 14.921 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.355 \\ 0.812 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with system states $[v\ r\ \psi\ \ Y]^T$, which corresponds to the linearization of the rigid body dynamics of a passenger vehicle. The first state, v, is the lateral velocity, the second state, r, is the yaw rate, the third state, ψ, represents the vehicle orientation and the fourth state, Y, is the lateral deviation from an intended lane position. The input to the system is the angular position of the front wheels relative to the chassis.

Transforming the system into the canonical form for design of the switching surface as in equation (12) yields

$$A_{11} = \begin{bmatrix} -3.9422 & 0 & 0 \\ 1 & 0 & -14.921 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 12.4066 \\ -1.6687 \\ 1 \end{bmatrix} \tag{30}$$

The gain from the LMI tool solver with $\delta = 0.2$, $\epsilon = 1$ and $M = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ yields

$$K = [\, -0.7579 \quad 8.1203 \,] \tag{31}$$

and hence

$$F = [\, 4.997 \quad 0.6154 \quad 0.4664 \,] \tag{32}$$

The sliding mode dynamics have poles at
$\{-1.7479 + 1.2162i \quad -1.7479 - 1.2162i \quad -9.8313\}$

The simulation of the states of the system with initial conditions $[0\ \ 0\ \ 1\ \ 0]$ is plotted in Fig. 1. The switching functions are shown in Fig. 2.

7. CONCLUSION

The development of sliding mode schemes for uncertain linear system representations when only output information is available to the controller has been presented. A descriptor Lyapunov functional method for switching function design has been derived. The LMI procedure can give desirable SOFSMC dynamics. The numerical example shows the effectiveness of the method.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge EPSRC support via research grant EP/E 02076311.

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