An Approach to Pole Placement Method with Output Feedback

Selda Guney*. Ayten Atasoy**

*Electrical-Electronics Engineering Department, Karadeniz Technical University, Trabzon, Turkey (*e-mail:seldakul@ktu.edu.tr, ** e-mail:ayten@ktu.edu.tr)*

Abstract: In this paper, we have presented a simple method in order to solve the pole placement problem of linear output feedback systems with state space models. Pole placement is a control method assigned to arbitrary closed loop poles by state or output feedback. In linear systems, poles have influence on stability, system response, transient response, and band width. Pole placement methods are used in the design of different control systems. This paper presents a numerical algorithm for pole placement with output feedback. Earlier, a method based on Sylvester's equation had been applied to pole placement with state feedback. In our work, this method is applied to pole placement with output feedback. It has been obtained by using generalised inverse approach. The pseudo-inverse C^+ of an m-by-n output matrix C caused a problem while output feedback matrix K_o is calculated. The generalised inverse approach is used for overcoming this problem. It is shown that real different poles and repeated poles are assigned via output feedback by using the given algorithm. The efficiency of the algorithm is denoted with several extensive numerical examples. Also the performance of the method is tested for different poles on various systems. The results are compared with generalised mapping approach.

Keywords: pole placement, output feedback system, linear time-invariant system

1. INTRODUCTION

One of the important results of pole placement methods with state feedback for linear time-invariant, and multi input-multi output systems was presented in Wonham (1967). In the current study, n, m, and l are the number of state variables, outputs and inputs, respectively. With some assumptions it had been shown that m closed loop poles which are independent from each other are assigned arbitrarily (Jameson, 1970). If the number of output variables is less than the order of the system, then it is always possible, by a constant feedback gain matrix, to assign m poles of the closed loop system matrix (Davison, 1970). Maximum (m, l) desired poles can be assigned by using output feedback (Davison and Chatterjee, 1971; Sridhor and Lindorff, 1972). If the system is controllable and observable, and if $n \leq m+l-1$, all set of distinct closed loop poles are assigned arbitrarily by output feedback gain (Kimura, 1975) and minimum(n, $m+l-1$) poles can be assigned arbitrarily by using output feedback presented at the same time (Davison and Wang, 1975). Kimura extended his work and showed that arbitrary pole placement is possible for all systems which satisfy $n< l+m+v-1$, $l>\mu$, $m\geq v$ where v and μ are so-called controllability and the observability indexes of the system, respectively (Kimura, 1977). Transition from state feedback to output feedback was done by using generalised matrix inverse (Munro and Vardulakis, 1973; Munro, 1974; Vardulakis, 1975). This method was applied to uncertain systems with generalised mapping approach (Soylemez, 1999).

2. POLE PLACEMENT

Consider a linear time-invariant multivariable system represented by the following equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

Here the state vector x, the input vector u and the output vector y are functions of a real variable t (time), with values in R^{nx1}, R^{lx1} and R^{mx1} respectively. The dot denotes the derivative with respect to t. A, B, and C are real matrices of sizes nxn, nxl and mxn. The poles of the system are the eigenvalues of A. (A, B) is controllable and (A, C) is observable.

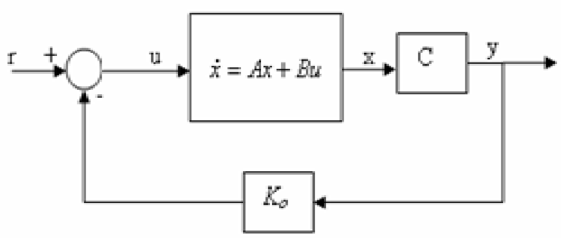

Fig. 1. Pole placement with output feedback

Pole placement with output feedback is shown in Fig. 1. In this study, the reference signal, r, is given zero value. If an output feedback control $u=-K_o y$ is applied to (1), the closed-loop system becomes

$$\dot{x}(t) = (A - BK_o C)x(t). \tag{3}$$

The poles assigned with output feedback are $\Lambda=\{\lambda_1, \lambda_2,...,\lambda_n\}$. The problem considered in this paper is finding K_o matrix for assigning the poles.

2.1 Generalised Matrix Inverse Approach

The closed-loop poles become the eigenvalues of $A\text{-}BK_oC$ and the closed-loop eigenvalues are denoted by:

$$\Lambda = eig(A - BK_o C) \tag{4}$$

If the state feedback control $u=-K_d x$ is used, the closed-loop poles become the eigenvalues of $A\text{-}BK_d$ and the closed-loop eigenvalues are denoted by:

$$\Lambda = eig(A - BK_d) \tag{5}$$

Therefore, (5) is derived from (3) and (4).

$$K_o = K_d C^+ \tag{6}$$

C^+ is the pseudo-inverse of C matrix in (6). In the generalised matrix inverse method; C matrix is partitioned into $C_1 \in R^{mxm}$ and $C_2 \in R^{mx(n-m)}$ sub matrices. All system is transformed by the similarity transformation matrix (Patel, 1974). The similarity transformation matrix is defined as indicated below:

$$T = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-m} \end{bmatrix} \tag{7}$$

The transformed system is defined with $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ matrices.

$$\widetilde{A} = T^{-1}AT \tag{8}$$

$$\widetilde{B} = T^{-1}B \tag{9}$$

$$\widetilde{C} = CT \tag{10}$$

By inserting equation (7) in (10), the following equation can be obtained.

$$C = \begin{bmatrix} I_m & 0 \end{bmatrix} \tag{11}$$

K_d, state feedback matrix, must satisfy (12) (Soylemez, 1999). Then also (13) is obtained.

$$BK_d \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix} = 0 \tag{12}$$

$$K_o = K_d \widetilde{C}^+ \tag{13}$$

If the above relation is valid, then the last (n-m) column of K_d must be zero. This requires arbitrary changes in K_d. But this is not possible for all algorithms. The only algorithms that make it possible are the ones whose parameters are chosen freely.

2.2 Output Feedback via Sylvester-Lyapunov Equations

In this study, the generalised Sylvester equation which is the most important feature of the method has been analysed for the pole placement with the output feedback (Munro and Vardulakis, 1973; Bhattacharya and De Sousa, 1982).

In order to obtain state feedback matrix, Sylvester equation is converted into Lyapunov equation. Then the state feedback matrix is transformed to the output feedback matrix by using generalised matrix inverse method.

The system given by (1) and (2) is converted by similarity transformation matrix to satisfy (12) and (13). The converted system is described as:

$$\begin{aligned} \dot{x} &= \widetilde{A}x + \widetilde{B}u \\ y &= \widetilde{C}x \end{aligned} \tag{14}$$

When Sylvester equation is applied to the system given by (14), (15) is obtained (Soylemez, 1999).

$$P\widetilde{A}_k - \widetilde{A}P = -\widetilde{B}M \tag{15}$$

$$M = K_d P \tag{16}$$

$$G = -\widetilde{B}M \tag{17}$$

The nxn square matrix $\widetilde{A}_k$, whose diagonal elements are equal to assigned eigenvalues, is in Jordan canonical form. Equation (15) is linearized by converting to Lyapunov equation. P transformation matrix is obtained by utilizing this linearization. The g vector is obtained from the columns of G matrix.

$$X_A = (-\widetilde{A} \otimes I_n) + (I_n \otimes \widetilde{A}_k') \tag{18}$$

$$p = X_A^{-1} g \tag{19}$$

The p vector given by (19) constitutes the columns P matrix. Thus state feedback matrix can be computed easily. Also output feedback matrix is obtained from state feedback matrix given by (6).

$$K_d = MP^{-1} \quad (20)$$

$$K_o = K_d \begin{bmatrix} I_m \\ 0 \end{bmatrix} \quad (21)$$

3. EXAMPLES

Example 1: The given algorithm in this work is applied to the following third-order system.

$$A = \begin{bmatrix} -11.4 & -3.5 & 0 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (22)$$

$$B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \quad (23)$$

$$C = \begin{bmatrix} 1 & 0 & 1.425 \\ 1 & -1 & 0 \end{bmatrix} \quad (24)$$

Assigned eigenvalues $\Lambda=\{-1,-2,-3\}$ are all different poles. At first, T transformation matrix given by (7) is derived. Then $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ matrices are found by transforming the whole system. The parameter matrix M is composed of symbolic variables and parameters which can be chosen arbitrarily according to dimensions of the system matrices.

$$M = \begin{bmatrix} m_1 & m_2 & 1 \\ -1 & 1 & -1 \end{bmatrix} \quad (25)$$

G matrix is converted to g column vector by solving (17).

$$G = \begin{bmatrix} -2m_1+1 & -2m_2-1 & -1 \\ -2m_1+2 & -2m_2-2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (26)$$

X_A is obtained from $\tilde{A}$ and $\tilde{A}_k$ by calculating Kronecker products as given in equation (18).

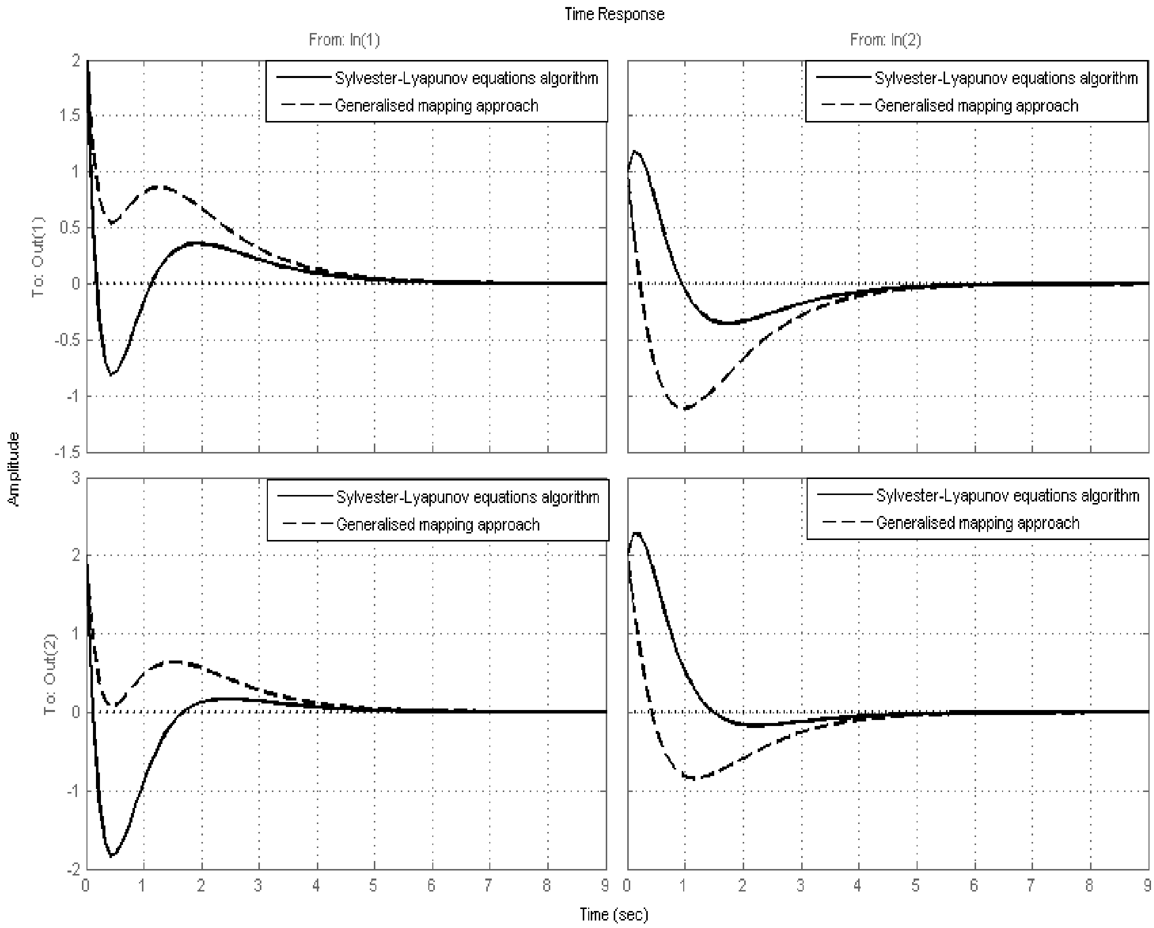

Fig. 2. Time response obtained from all inputs to all outputs when Sylvester-Lyapunov equations algorithm and generalized mapping approach are applied to the system given by example 1.

$$X_A=\begin{bmatrix}10.48 & 0 & 0 & -2.08 & 0 & 0 & -19.2 & 0 & 0\\ 0 & 11.48 & 0 & 0 & -2.08 & 0 & 0 & -19.2 & 0\\ 0 & 0 & 12.48 & 0 & 0 & -2.08 & 0 & 0 & -19.2\\ 18.9 & 0 & 0 & -6.5 & 0 & 0 & -26.9 & 0 & 0\\ 0 & 18.9 & 0 & 0 & -5.5 & 0 & 0 & -26.9 & 0\\ 0 & 0 & 18.9 & 0 & 1 & -4.5 & 0 & 0 & -26.9\\ -1 & 0 & 0 & 1 & 0 & 0 & -1.57 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & -0.57 & 0\\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0.425\end{bmatrix} \quad (27)$$

The p column vector is solved by inserting (26) and (27) in (19). Thus, P matrix is obtained.

$$P=\begin{bmatrix}-0.875m_1-0.23 & 1.11-2.02m_2 & 6.95\\ -0.4375-1.25m_1 & 1.437-2.5m_2 & 5.25\\ -0.238m_1-0.13 & 0.562-0.833m_2 & 4\end{bmatrix} \quad (28)$$

The state feedback matrix K_d is obtained from (20) depending on m_1 and m_2 which are symbolic variables. Symbolic variables which must satisfy (12) are found and inserted in the state feedback matrix.

$$K_d=\begin{bmatrix}2.7827 & -3.4933 & 0\\ 2.1837 & -3.0812 & 0\end{bmatrix} \quad (29)$$

Also output feedback matrix is found as:

$$K_o=\begin{bmatrix}2.7827 & -3.4933\\ 2.1837 & -3.0812\end{bmatrix} \quad (30)$$

When the output feedback matrices obtained from generalised mapping approach and Sylvester-Lyapunov equations are applied to example 1, all outputs as shown in Fig. 2. As seen in Fig. 2, the system is stable from all inputs to all outputs for two algorithms. Reference signal is zero and is applied to all inputs.

Example 2: The same algorithm is also applied to the following system. Assigned eigenvalues are Λ={-3,-3,-4}. The system has two repeated eigenvalues in -3.

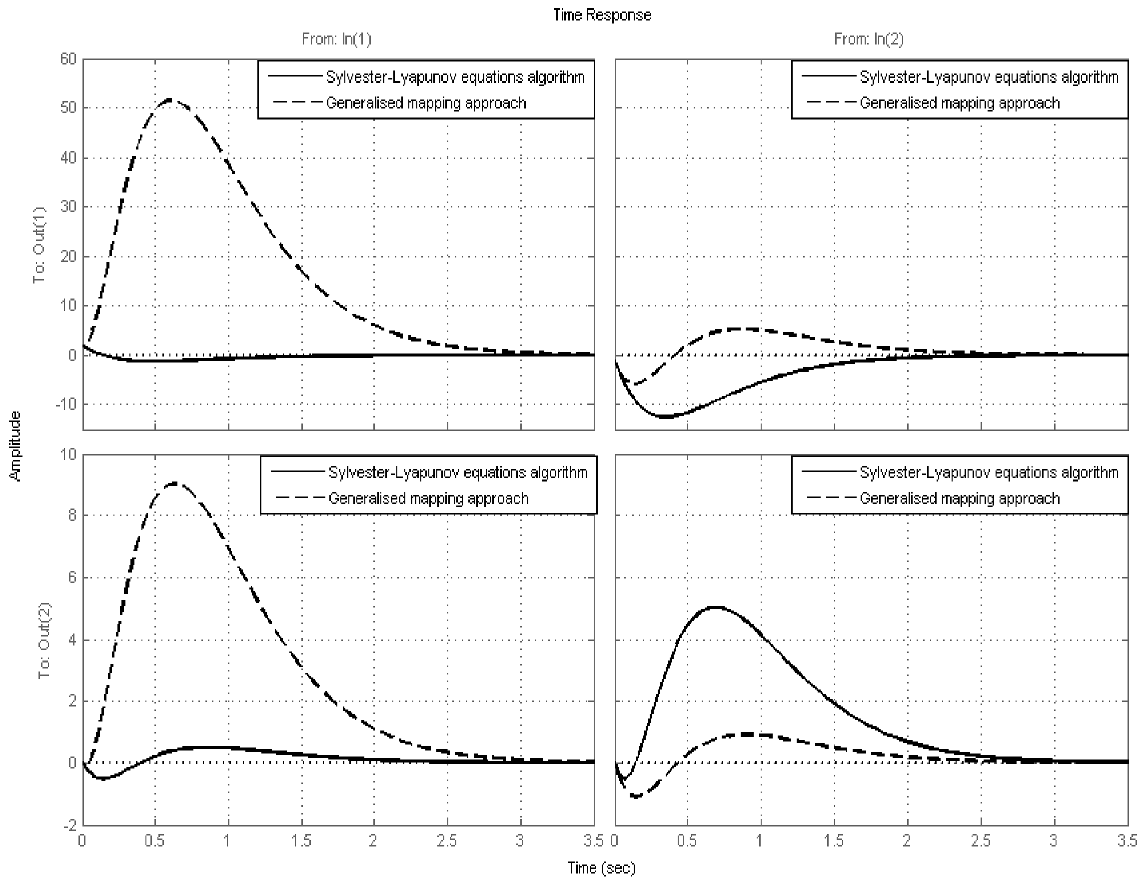

Fig. 3. Time response obtained from all inputs to all outputs when Sylvester-Lyapunov equations algorithm and generalized mapping approach are applied to the system given by example 2.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 19.62 & 0 & -8.86 \\ 0 & 0 & -100 \end{bmatrix} \quad (31)$$

$$B = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (32)$$

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \quad (33)$$

The output feedback matrix is obtained as shown below:

$$K_o = \begin{bmatrix} -44.7401 & -11.3932 \\ 0.5199 & -0.1689 \end{bmatrix} \quad (34)$$

It is seen that the system is stable from all inputs to outputs (Fig. 3). When the output feedback matrices obtained from generalised mapping approach and Sylvester-Lyapunov equations are applied to example 2, all outputs as shown in Fig. 3. Here reference signal is zero and is applied to all inputs. It is seen that the steady-state values of two methods are considerably close when they are compared with each other. As in Fig. 3, their settling time is the same and both of the systems are stable. Also, the time response exhibits small overshoot and undershoot in output feedback via Sylvester-Lyapunov equations method when it is compared with the generalised mapping approach.

4. CONCLUSIONS

In the current study, one of the pole placement methods with state feedback is generalised for output feedback algorithm. A different approach to pole placement algorithm based on the generalised matrix inverse and the Sylvester equation is derived. This algorithm is tested for systems with different and repeated poles. The results are compared with the generalised mapping approach. It is seen that desired poles are assigned successfully by using output feedback matrix K_o obtained with the algorithm given in this study. This method can be specifically suggested for use with small dimension systems because if the dimension of algorithm increases, the computations load increases too.

REFERENCES

Bhattacharyya, S.P. and E. De Sousa (1982). Pole Assignment via Sylvester's Equation. *System Control Letter*, **1**, 261-263.

Davison, E.J. (1970). On Pole Assignment in Linear Systems with Incomplete State Feedback. *IEEE Transactions on Automatic Control*, **15**, 348-351.

Davison, E.J. and R. Chatterjee (1971). A Note on Pole Assignment in Linear Systems with Incomplete State Feedback. *IEEE Transactions on Automatic Control*, **16**, 98-99.

Davison, E.J. and S.H. Wang (1975). On Pole Assignment in Linear Multivariable Systems Using Output Feedback. *IEEE Transactions on Automatic Control*, **15**, 516-518.

Jameson, A. (1970). Design of a Single-Input System for Specified Roots Using Output Feedback. *IEEE Transactions on Automatic Control*, **15**, 345-348.

Kimura, H. (1975). Pole Assignment by Gain Output Feedback. *IEEE Transactions on Automatic Control*, **20**, 509-516.

Kimura, H. (1977). A Further Result on the Problem of Pole Assignment by Output Feedback. *IEEE Transactions on Automatic Control*, **22**, 458-463.

Munro, N. (1974). Further Results on Pole-shifting Using Output Feedback. *International Journal of Control*, **20**, 775-786.

Munro, N. and A. Vardulakis (1973). Pole-shifting Using Output Feedback. *International Journal of Control*, **18**, 1267-1273.

Patel, R.V. (1974). On Output Feedback Pole Assignability. *International Journal of Control*, **20**, 955-959.

Soylemez, M.T. (1999). *Pole Assignment for Uncertain Systems*. 5, Research Studies Press Ltd, Hetfordshire.

Sridhar, B. and D.P. Lindorff (1972). A Note on Pole Assignment. *IEEE Transactions on Automatic Control*, **18**, 822-823.

Vardulakis, A.I. (1975). A Sufficient Condition for n Specified Eigenvalues to be Assigned Under Constant Output Feedback. *IEEE Transactions on Automatic Control*, **20**, 428-429.

Vardulakis, A.I. (1976). Generalized Root Locus Assignment of All Poles of a Multivariable System by Output Feedback. *International Journal of Control*, **23**, 39-47.

Wonham, W.M. (1967). On pole assignment in multi-input controllable linear systems. *IEEE Transactions on Automatic Control*, **12**, 660-665.