Poiseuille Flow Controller Design via the Method of Inequalities

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Abstract: This paper investigates the use of the Method of Inequalities (MoI) to design output-feedback compensators for the problem of the control of laminar plane Poiseuille flow. In common with many flows, the dynamics of plane Poiseuille flow are very non-normal. Consequently, small perturbations grow rapidly with a large transient that may trigger nonlinearities and lead to turbulence even though such perturbations would, in a linear flow, eventually decay. Such a system can be described as a *conditionally linear system*. The sensitivity is measured using the maximum transient energy growth, which is widely used in the fluids dynamics community. The paper considers two approaches. In the first, the MoI is used to design low-order proportional and P+D controllers. In the second approach, the MoI is combined with McFarlane and Glover's $\mathcal{H}_\infty$ loop-shaping design procedure in a mixed-optimization approach. The results show that the low-order controllers do reduce the maximum transient energy growth but the reduction is not satisfactory. Furthermore, the $\mathcal{H}_\infty$ approach does not improve the performance.

Keywords: Transient energy growth, transient behaviour, flow control, Poiseuille flow, Method of Inequalities (MoI), mixed optimization, $\mathcal{H}_\infty$-optimization.

1. INTRODUCTION

The problem of stabilizing fluid flows by feedback control has recently become a topic of much interest [e.g. Scott Collis et al., 2004]. Fluid flow dynamics are often highly non-normal — that is their eigenvectors are closely aligned — and this non-normality is one factor that makes fluid systems hard to control. Traditionally, fluid dynamicists have assessed the stability of systems using Lyapunov's first method, paying little attention to the eigenvectors and the system sensitivity. For non-normal flows this leads to difficulties in resolving the differences between measured and predicted flow stability [Baggett et al., 1995]. Plane Poiseuille or channel flow is the unidirectional flow between two infinite parallel planes. This flow is laminar and stable for low Reynolds numbers, but at high Reynolds numbers the flow becomes unstable resulting in turbulence. Experiments show that the flow undergoes transition to turbulence for Reynolds number as low as 1000 [Carlson et al., 1982]. However, eigenvalue predictions show the flow to be stable at Reynolds numbers below approximately 5772 [Orszag, 1971]. The non-normal nature of the dynamics makes the flow very sensitive. Hence an initial perturbation will grow to very large values before decaying. This can drive the system into regions where the non-linearities are significant and trigger turbulence.

The system dynamics can thus be considered as conditionally linear [Zakian, 1979]. That is, the system can be considered linear provided the state remains within some region of the state space that is sufficiently close to the steady state. In this paper, we explicitly consider the energy of perturbations (or transient energy) which is a measure of the size of the perturbations of the state. It has a clear physical meaning and is a fundamental notion in the study of turbulence and transition. Consequently, the maximum transient energy growth following some energy-bounded initial state perturbation is often used as a performance measure for fluid flow systems [e.g. Schmid and Henningson, 2001, Bewley and Liu, 1998].

The problem of controlling plane Poiseuille flow has received some study. For example, optimal linear quadratic methods have been considered by Bewley and Liu [1998] and by McKernan et al. [2007]. In these papers, the maximum transient energy growth is considered in the analysis of the design, but not explicitly in the design formulation. State feedback control that can minimise an upper bound on the maximum transient energy growth for plane Poiseuille flow is considered in Whidborne et al. [2008]. However, state feedback is not available for fluid flow systems. The actual maximum transient energy growth can, in principle, be minimized for the output feedback problem [Whidborne and McKernan, 2007], but the method is too computationally expensive for the Poiseuille flow problem. Furthermore it results in extremely high order controllers. Thus this paper considers two approaches that use the Method of Inequalities (MoI). In the first, the MoI is used to design low-order controllers, namely proportional and P+D controllers. In the second approach, the MoI is combined with McFarlane and Glover's $\mathcal{H}_\infty$ loop-shaping design procedure in a mixed-optimization approach.

A summary of the derivation of the state-space model is presented in the next section. This material can also be found in Whidborne et al. [2008], but is reproduced here for convenience. A more detailed exposition can be found in McKernan et al. [2006], and full details in McKernan

[2006]. Section 3 defines the maximum transient energy growth and proposes that the control signal magnitude is measured in a similar way. The MoI and mixed optimization are introduced in Sections 4 and 5 respectively. The design results are given in Section 6. Finally, comments and conclusions are provided.

2. PLANE POISEUILLE FLOW

Incompressible fluid flow is described by the Navier-Stokes and continuity equations. The Navier-Stokes equations

$$\dot{\boldsymbol{U}} + (\boldsymbol{U} \cdot \nabla)\, \boldsymbol{U} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2 \boldsymbol{U} \tag{1}$$

form a set of three coupled, non-linear, partial differential equations representing conservation of momentum where $\boldsymbol{U}$ is velocity, P is pressure, ρ is density (uniform) and μ is viscosity (uniform), and the continuity equation

$$\nabla \cdot \boldsymbol{U} = 0 \tag{2}$$

is an additional constraint representing the conservation of mass.

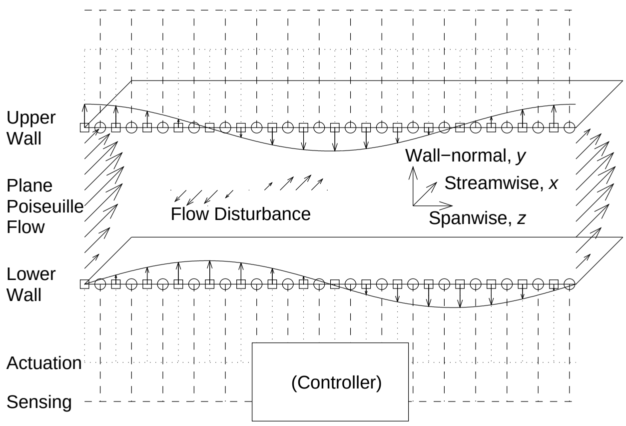

Fig. 1. Plane Poiseuille flow.

Laminar Poiseuille flow, shown in Figure 1, has a parabolic streamwise velocity profile, with no slip occurring at the bounding parallel planes. It undergoes transition to turbulence when small perturbations $\boldsymbol{u} = (u, v, w)$, p about the steady base profile, $\boldsymbol{U}_b = ((1-(y/h)^2)U_{cl}, 0, 0)$, P_b, grow spatially and temporally to form a self-sustaining turbulent flow. The Navier-Stokes equations for the perturbations about the base flow, $\boldsymbol{U}_b$, become

$$\dot{\boldsymbol{u}} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} + (\boldsymbol{U}_b \cdot \nabla)\, \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{U}_b = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \boldsymbol{u} \tag{3}$$

Assuming the perturbations are small compared to the base flow, the second-order nonlinear term, $(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}$, can be discarded. Non-dimensionalizing (3) by dividing length scales by the channel half-height h, dividing velocities by the base centreline velocity U_{cl} and dividing pressure by ρU_{cl}^2, gives

$$\dot{\boldsymbol{u}} + (\boldsymbol{U}_b \cdot \nabla)\, \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{U}_b = -\nabla p + \frac{1}{R}\nabla^2 \boldsymbol{u} \tag{4}$$

where $R := \rho U_{cl} h/\mu$ is the dimensionless Reynolds number. Being linear, the continuity equation, (2), simply becomes

$$\nabla \cdot \boldsymbol{u} = 0 \tag{5}$$

The state of the flow can be determined from wall shear stress and pressure measurements, and the flow can be influenced by the manipulation of the conditions on its boundaries, for example by wall transpiration, which is the injection and suction of fluid at the walls. Hence active feedback control of the evolution of transition is feasible. The proposed scheme is shown in Figure 1. However, (4) and (5) are infinite dimensional, so in order to be able to use standard finite dimension control methods, and to ensure that the controller is practically implementable, they must be approximated by a finite dimension linear time-invariant system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{6}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{7}$$

However, a straightforward discretisation results in a descriptor system with the form $\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ where $\mathbf{E}$ is singular. This is a consequence of the algebraic constraint imposed by the continuity equation, (5), that does not contain pressure.

To proceed, the pressure perturbation term is eliminated from (4) by substituting (5) giving an expression for the wall-normal velocity

$$\frac{\partial(\nabla^2 v)}{\partial t} + U_b \frac{\partial(\nabla^2 v)}{\partial x} - \frac{\partial^2 U_b}{\partial y^2}\frac{\partial v}{\partial x} - \frac{1}{R}\nabla^2(\nabla^2 v) = 0 \tag{8}$$

To completely describe a three dimensional flow perturbation, a second equation is required to describe the wall-normal vorticity, η, where

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{9}$$

and (4) and (5) give

$$\frac{\partial \eta}{\partial t} + \frac{\partial U_b}{\partial y}\frac{\partial v}{\partial z} + U_b \frac{\partial \eta}{\partial x} - \frac{1}{R}\nabla^2 \eta = 0 \tag{10}$$

To implement control by wall transpiration, the no-slip wall boundary conditions at $y = \pm 1$ are replaced by prescribed wall transpiration velocities, $(u(\pm 1) = 0, v(\pm 1) \neq 0, w(\pm 1) = 0)$. It is assumed that disturbances on $\boldsymbol{u}$ vary in the streamwise (x), wall-normal (y), and spanwise (z) directions. Variations in the wall-normal direction are assumed to be non-periodic and are represented by a modified Chebyshev series that fulfils the wall boundary conditions. Variations in the streamwise and spanwise directions are assumed to have a periodic representation, $\Re(e^{i(\alpha x + \beta z)})$, so flow disturbances grow in time, but not in space. The terms α and β are the streamwise and spanwise wave numbers respectively. Substituting the assumed solutions into (8) and (10), and assuming an exponential time variation results in the classical Orr-Sommerfield and Squire equations respectively.

After some manipulation of the equations, boundary control of the linearised Navier-Stokes equations in a channel at a particular wave number pair, (α, β) (with associated variables denoted by $\tilde{u}$, $\tilde{v}$, etc), can be represented as a linear state-space system in the standard form of (6). The linearised Navier-Stokes equations are evaluated at N locations in the wall-normal direction (with the locations more closely spaced near the walls), and the state variables $\mathbf{x}$ are the Chebyshev coefficients of the wall-normal velocity, $\tilde{v}$, and vorticity, $\tilde{\eta}$, perturbations concatenated with the upper and lower wall $\tilde{v}$ transpiration velocities. For details, see McKernan [2006].

The inputs, $\mathbf{u}$, are the rates of change of symmetrical and antisymmetrical components of wall transpiration velocity. Since these are rates of change, the system contains two integrators, with eigenvectors representing symmetrical and antisymmetrical steady-state transpiration from the walls. The measurements are the wall shear-stress Fourier coefficients on the upper and lower walls. The Chebyshev coefficients are complex, but the state-space system is made real-valued by decomposing it into its real- and imaginary-valued parts. The test case considered here is $\alpha = 0$, $\beta = 2.044$, $R = 5000$. This test case is linearly stable but has the largest transient energy growth over all unit initial conditions, time and wave-number pairs, and represents the very earliest stages of the transition to turbulence. The model is discretised in the wall-normal direction with $N = 20$. The order of the resulting model is $2N - 2$. A 38th order plant model is high for control purposes, but errors become significant at lower values of N [McKernan, 2006]. Modelling the turbulence itself would involve using many more degrees of freedom.

3. TRANSIENT ENERGY GROWTH

Consider the asymptotically stable linear time-invariant system described by the initial value problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{11}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{x}_0 \in \mathbb{R}^n$ which has the continuous solution $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}^n$, $t \mapsto \mathbf{\Phi}(t)\mathbf{x}_0$, where $\mathbf{\Phi}(t)$ is the state transition matrix given by

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \sum_{i=0}^{\infty} \mathbf{A}^i t^i / i!. \tag{12}$$

For simplicity of presentation in this section, the transient energy, $\mathcal{E}(t)$, is defined as

$$\mathcal{E}(t) := \max\left\{\|\mathbf{x}(t)\|^2 : \|\mathbf{x}_0\| = 1\right\}. \tag{13}$$

The maximum transient energy growth, $\hat{\mathcal{E}}$, is defined as

$$\hat{\mathcal{E}} := \max\left\{\mathcal{E}(t) : t \geq 0\right\}. \tag{14}$$

In fluid dynamic practice, the transient energy, $\mathcal{E}(t)$, is

$$\mathcal{E}(t) = \max\left\{\|\mathbf{W}\mathbf{x}(t)\|^2 : \|\mathbf{W}\mathbf{x}(0)\| = 1\right\}, \tag{15}$$

where $\mathbf{W} > 0$ is a constant weight. Hence, for the remaining results in this section to be applicable to the laminar plane Poiseuille flow problem, a simple change of variables $\tilde{\mathbf{x}} = \mathbf{W}\mathbf{x}$ should be performed.

Now consider the linear time-invariant plant

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{16}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{n\times \ell}$, $\mathbf{u}(t) \in \mathbb{R}^\ell$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{y}(t) \in \mathbb{R}^m$ with feedback controller

$$\begin{aligned} \dot{\mathbf{x}}_k(t) &= \mathbf{A}_k\mathbf{x}_k(t) + \mathbf{B}_k\mathbf{y}(t), \quad \mathbf{x}_k(0) = \mathbf{x}_{k0}, \\ \mathbf{u}(t) &= \mathbf{C}_k\mathbf{x}_k(t) + \mathbf{D}_k y(t), \end{aligned} \tag{17}$$

with $\mathbf{A}_k \in \mathbb{R}^{n_k\times n_k}$, $\mathbf{B}_k \in \mathbb{R}^{n_k\times m}$, $\mathbf{C}_k \in \mathbb{R}^{\ell\times n_k}$, $\mathbf{D}_k \in \mathbb{R}^{\ell\times m}$. The closed loop system is given by

$$\dot{\mathbf{x}}_c(t) = \mathbf{A}_c\mathbf{x}_c(t), \quad \mathbf{x}_c(0) = \mathbf{x}_{c0} \tag{18}$$

where

$$\mathbf{A}_c := \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{D}_k\mathbf{C} & \mathbf{B}\mathbf{C}_k \\ \mathbf{B}_k\mathbf{C} & \mathbf{A}_k \end{bmatrix}, \quad \mathbf{x}_c(t) := \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_k(t) \end{bmatrix}. \tag{19}$$

The maximum transient energy growth of the plant is

$$\hat{\mathcal{E}} = \max\left\{\|\mathbf{x}(t)\|^2 : \|\mathbf{x}_0\| = 1,\, \mathbf{x}_{k0} = 0,\, t \geq 0\right\} \tag{20}$$

and can be evaluated by $\|\Phi_c(t)\|^2$ where

$$\Phi_c(t) := [I_n\ 0_{n_k}]e^{\mathbf{A}_c t}[I_n\ 0_{n_k}]^T. \tag{21}$$

and $\|\cdot\|$ denotes the spectral norm. The transient energy in the controller states is irrelevant to the problem under consideration because the controller can always be normalized by appropriate choice of controller state basis. This is the problem of controller state scaling [see Ahmed and Belanger, 1984, for example].

In order to limit the amount of effort generated by the controller in a closed loop system, the maximum control "transient energy growth" is defined as

$$\hat{\mathcal{U}} := \max\left\{\|\mathbf{u}(t)\|^2 : \|\mathbf{x}_0\| = 1,\, \mathbf{x}_{k0} = 0,\, t \geq 0\right\}. \tag{22}$$

4. THE METHOD OF INEQUALITIES(MOI)

In the MoI [Zakian and Al-Naib, 1973], the control design problem is expressed as a set of algebraic inequalities that need to be satisfied for a successful design. The design problem is expressed as

$$\phi_i(\mathbf{p}) \leq \epsilon_i \text{ for } i = 1 \ldots n, \tag{23}$$

where ϵ_i are real numbers, $\mathbf{p} \in \mathcal{P}$ is a real vector $(p_1, p_2, \ldots, p_q)$ chosen from a given set $\mathcal{P}$ and ϕ_i are real functions of $\mathbf{p}$. The design goals ϵ_i are chosen by the designer and represent the largest tolerable values of the objective functions ϕ_i. The aim of the design is to find a $\mathbf{p}$ that simultaneously satisfies the set of inequalities, such a point is known as an *admissible point*.

A solution to the set of inequalities, (23), is obtained by means of numerical search algorithms. Generally, the design process is interactive, with the computer providing information to the designer about conflicting design requirements, and the designer adjusting the inequalities to explore the various possible solutions to the problem. The progress of the search algorithm should be monitored, and, if a solution is not found, the designer may either change the starting point, relax some of the design goals ϵ or change the design configuration. Alternatively, if a solution is found easily, to improve the quality of the design, the design goals could be tightened or additional design objectives could be included in (23). The design process is thus a two way process, with the MoI providing information to the designer about conflicting design requirements, and the designer making decisions about the 'trade-offs' between design requirements based on this information as well as on the designer's knowledge, experience and intuition about the particular problem. Further details on numerical algorithms can be found in Zakian and Al-Naib [1973], Whidborne et al. [1995a], Fonseca and Fleming [1998], Zakian [2005].

The functions $\phi_i(\mathbf{p})$ are typically functionals of the system step response, for example the rise-time, overshoot or the integral absolute error, or functionals of the frequency response, such as the bandwidth. For the Poiseuille flow control problem they are taken as the maximum transient energy growth and the maximum control transient energy growth. To ensure these are finite, the system must be closed-loop stable and so a measure of the system stability needs to be included. One suitable measure is the maximum real part of the closed-loop eigenvalues

$$\alpha_0 = \max_i \{\Re(\lambda_i(\mathbf{A}_c))\} \tag{24}$$

where $\{\lambda_i\}$ represents the set of eigenvalues of the closed-loop system matrix, $\mathbf{A}_c$. Generally, the design parameter, $\mathbf{p}$, parameterizes a controller with a particular structure [e.g. Liu et al., 2002] rather than the state space structure of (17). For example, $\mathbf{p} = (p_1, p_2)$ could parameterize a P+I controller $p_1 + p_2/s$. Because state space realizations are over-parameterized, this ensures a smaller dimension of the search space, $\mathcal{P}$.

The Poiseuille flow control problem can be formulated as follows:

Problem 1. Find a $\mathbf{p} \in \mathcal{P}$ and hence a $K(p)$ such that

$$\alpha_0(\mathbf{p}) \le \epsilon_\alpha, \tag{25}$$

$$\hat{\mathcal{E}}(\mathbf{p}) \le \epsilon_{\hat{\mathcal{E}}}, \tag{26}$$

$$\hat{\mathcal{U}}(\mathbf{p}) \le \epsilon_{\hat{\mathcal{U}}}, \tag{27}$$

where ϵ_α, $\epsilon_{\hat{\mathcal{E}}}$, and $\epsilon_{\hat{\mathcal{E}}}$ are prescribed tolerable values of α_0, $\hat{\mathcal{E}}$, and $\hat{\mathcal{U}}$ respectively.

5. MIXED OPTIMIZATION

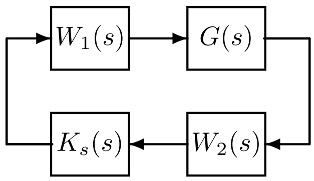

Fig. 2. Controller configuration for LSDP

The MoI can be combined with analytical optimization techniques in a mixed optimization approach by using the parameters of the weighting functions generally required by such techniques as the design parameters [Whidborne et al., 1994, 1995b, Postlethwaite et al., 1994]. Here, we use McFarlane and Glover's loop-shaping design procedure (LSDP) [McFarlane and Glover, 1990].

The LSDP maximizes robust stability to perturbations on the normalized coprime factors of a plant weighted by pre- and post-compensators $W_1(s)$ and $W_2(s)$ as shown in Figure 2. An explicit controller $K(s)$ for optimal γ

$$\gamma_0 = \inf_K \left\| \begin{bmatrix} W_1^{-1} K \\ W_2 \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} W_2^{-1} & GW_1 \end{bmatrix} \right\|_\infty \tag{28}$$

can be synthesized, the weights having been simply incorporated into the optimal controller $K_s(s)$ so that $K = W_1 K_s W_2$.

The problem can be formulated as the MoI as follows:

Problem 2. For the system of Figure 2, find a $\mathbf{p} \in \mathcal{P}$ and hence a $(W_1, W_2)(\mathbf{p})$ and K such that

$$\gamma_0(\mathbf{p}) \le \epsilon_\gamma, \tag{29}$$

$$\hat{\mathcal{E}}(\mathbf{p}) \le \epsilon_{\hat{\mathcal{E}}}, \tag{30}$$

$$\hat{\mathcal{U}}(\mathbf{p}) \le \epsilon_{\hat{\mathcal{U}}}, \tag{31}$$

where $(W_1, W_2)(\mathbf{p})$ is a pair of fixed order weighting functions with real parameters $\mathbf{p} = (p_1, p_2, \ldots, p_q)$ and ϵ_γ, $\epsilon_{\hat{\mathcal{E}}}$, and $\epsilon_{\hat{\mathcal{U}}}$ are prescribed tolerable values of γ_0, $\hat{\mathcal{E}}$, and $\hat{\mathcal{U}}$ respectively.

6. RESULTS

6.1 Open loop

The transient energy $\mathcal{E}(t)$ of the linearized system with no wall transpiration control is shown in Figure 3. The maximum transient energy growth is $\hat{\mathcal{E}} = 4941$.

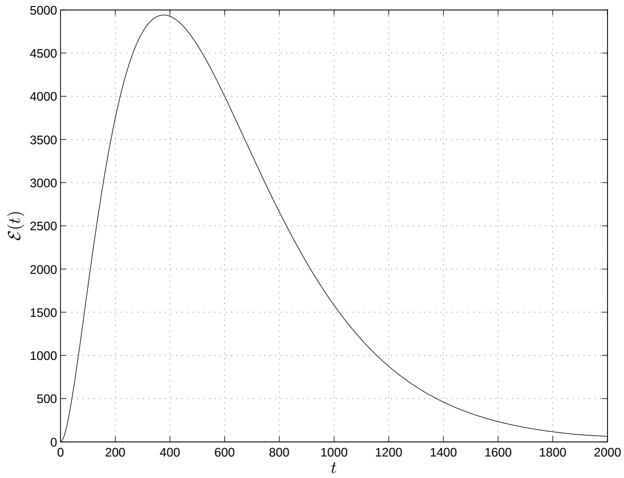

Fig. 3. Transient energy for open-loop system.

6.2 Low order controllers

The design goals are set at

$$\epsilon_\alpha = -1 \times 10^{-5}, \tag{32}$$

$$\epsilon_{\hat{\mathcal{E}}} = 1000, \tag{33}$$

$$\epsilon_{\hat{\mathcal{U}}} = 10. \tag{34}$$

Several low-order structures were tried, but no controller was found that satisfied Problem 1. After a small number of iterations, the proportional controller

$$\mathbf{K} = \begin{bmatrix} 0.7474 & 0.8655 & 0.3259 & -0.7862 \\ 0.7855 & 0.7855 & 0.2061 & 0.7757 \end{bmatrix} \tag{35}$$

was obtained with a performance

$$\alpha_0 = -1.7750 \times 10^{-3}, \tag{36}$$

$$\hat{\mathcal{E}} = 2781.4, \tag{37}$$

$$\hat{\mathcal{U}} = 9.7519. \tag{38}$$

Further iteration resulted in only a small improvement. The transient energy $\mathcal{E}(t)$ is shown in Figure 4, and the control transient energy is shown in Figure 5. Detail of the control transient energy is shown in Figure 6.

The P+D controller structure

$$K(s) = \mathbf{K}_p + \mathbf{K}_d \left(\frac{s}{s+a} \right) \tag{39}$$

was tried and after some iteration a controller with $a = 14.9058$ and

$$\mathbf{K}_p = \begin{bmatrix} 0.6151 & 0.4184 & 1.0055 & 0.4233 \\ 1.4066 & 0.0563 & -0.0266 & 0.3148 \end{bmatrix} \tag{40}$$

$$\mathbf{K}_d = \begin{bmatrix} 26.2916 & 42.0612 & 4.3891 & 4.8970 \\ 13.1916 & 92.8310 & 4.1459 & 13.3138 \end{bmatrix} \tag{41}$$

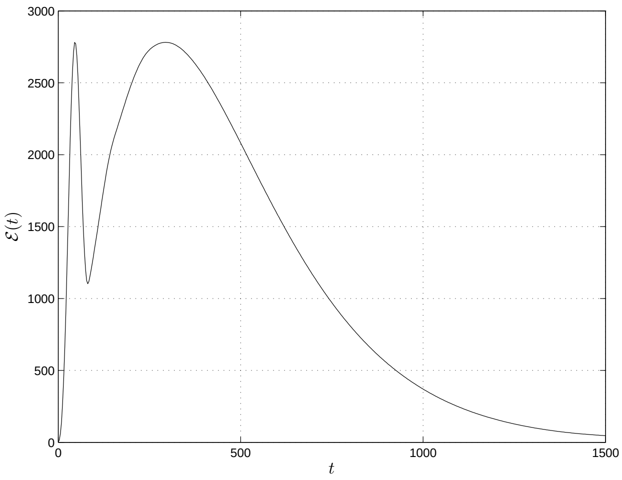

Fig. 4. Transient energy for proportional controller.

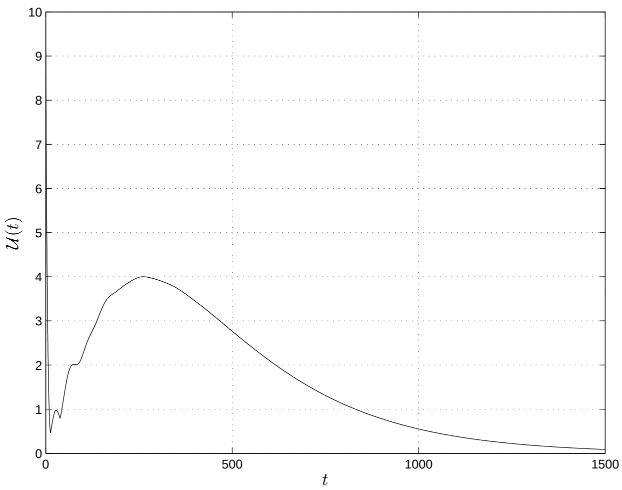

Fig. 5. Control transient energy for proportional controller.

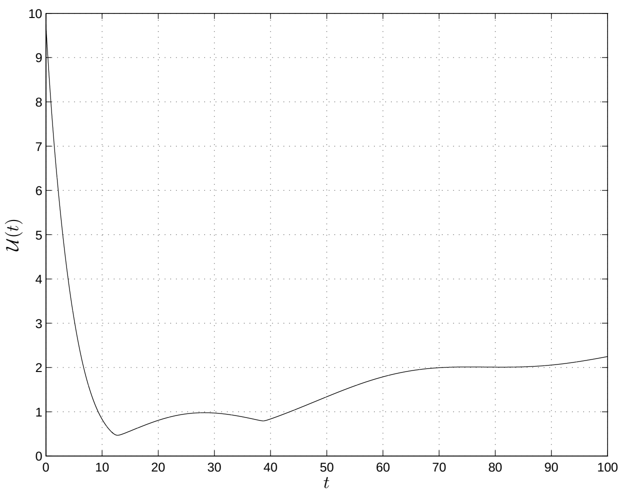

Fig. 6. Control transient energy for proportional controller (detail).

was obtained that gave a performance

$$\alpha_0 = -1.7806 \times 10^{-3}, \tag{42}$$
$$\hat{\mathcal{E}} = 2737.4 \tag{43}$$
$$\hat{\mathcal{U}} = 8.913. \tag{44}$$

This is only a small improvement on the proportional controller. The transient energy $\mathcal{E}(t)$ is shown in Figure 7 and is quite similar to Figure 4, the proportional controller case.

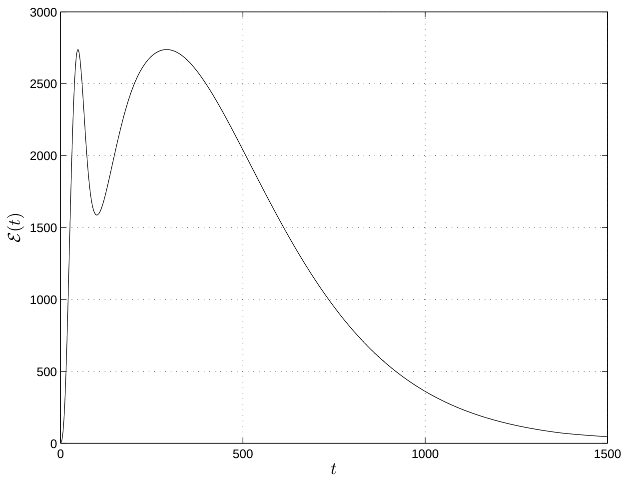

Fig. 7. Transient energy for P+D controller.

6.3 Mixed optimization

The design goals are set for Problem 2 at

$$\epsilon_\gamma = 5, \tag{45}$$
$$\epsilon_{\hat{\mathcal{E}}} = 1000, \tag{46}$$
$$\epsilon_{\hat{\mathcal{U}}} = 10. \tag{47}$$

The weighting function $W_1(s)$ was tried with a diagonal P+I structure and a diagonal P+D structure, but a simple proportional diagonal structure $W_1 = \mathrm{diag}([p_1, p_2])$ was found to be best. The post-plant weighting was set to be the identity $W_2 = I$. The resulting controller performance was fairly insensitive to the values of $\mathbf{p}$, thus $W_1 = I$ was chosen. The design goal, $\epsilon_{\hat{\mathcal{E}}}$ was not satisfied, and interestingly, the best value of $\hat{\mathcal{E}}$ was marginally greater than that for the proportional controller. The resulting performance was

$$\gamma_0 = 1.818, \tag{48}$$
$$\hat{\mathcal{E}} = 2884.1, \tag{49}$$
$$\hat{\mathcal{U}} = 8.876. \tag{50}$$

The transient energy $\mathcal{E}(t)$ is shown in Figure 8, and the control transient energy is shown in Figure 9.

7. CONCLUSIONS

In this paper, the design of both low order controllers and $\mathcal{H}_\infty$-optimal controllers using the MoI has been performed. The state feedback controllers obtained in Whidborne et al. [2008] gave a value of $\hat{\mathcal{E}} = 883$. This is considerably lower than the best value obtained in this study of $\hat{\mathcal{E}} = 2737$. The optimal $\mathcal{H}_\infty$ was unable to improve on the P+D controller and was actually marginally greater. This indicates that perhaps, for this system, there is little

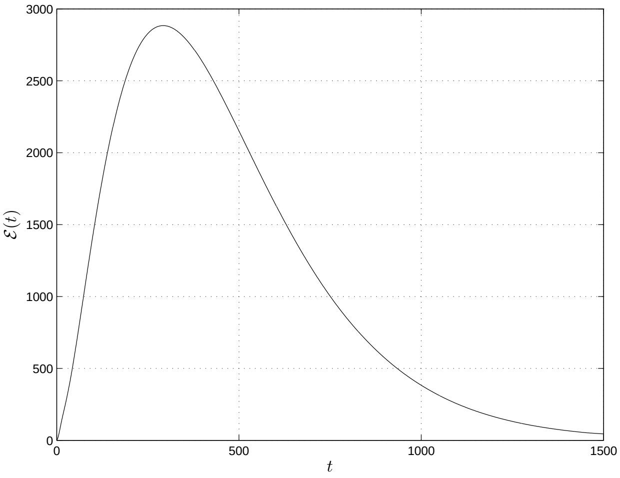

Fig. 8. Transient energy for LSDP controller.

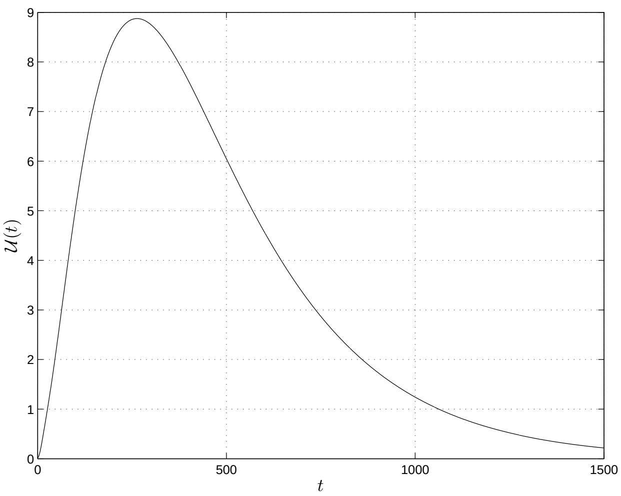

Fig. 9. Control transient energy for LSDP controller.

advantage in dynamic controllers and that the output feedback problem as posed here does not have any solutions. With an increase in computational power, the convex optimization methods of Whidborne and McKernan [2007] could be applied to this problem and determine what is the minimal $\hat{\mathcal{E}}$ and whether a solution exists. The problem of model order reduction for the maximum transient energy growth problem would aid the design of controllers, but that is a topic for further study.

Note that the design low-order controllers for the problem of plane Poiseuille flow has been considered before [Joshi et al., 1997] but but not the explicit consideration of the maximum transient energy growth. Similarly, Macfarlane and Glover's LSDP has been used for a spatially growing channel flow [Baramov et al., 2004], but transient energy growth was also not explicitly considered.

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