$\mathcal{H}_\infty$-based model order reduction using LMIs

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Abstract: In this paper new sufficient conditions are presented for the existence of a Lyapunov pair with a coupling rank constraint within a $\mathcal{H}_\infty$ minimization framework derived using the bounded real lemma and the projection lemma. The conditions are then used to propose a Linear Matrix Inequality (LMI) sub-optimal method to solve the model order reduction problem of general non-square LTI systems with a prescribed number of states to be removed. This alleviates the need for trace or rank minimization, iterations, or a priori choice of any new additional variable. The effectiveness and stability of the proposed LMI method is demonstrated by applications to several model order reduction problems.

1. INTRODUCTION

Advanced high-order optimal control methods, faced with the practical benefits of low-order system, has resulted in a recurrent need to be able to simplify a given dynamical system $\mathbf{G}(s) \in \mathcal{R}_q^{n\times m}(s)$, with one of a reduced order $\hat{\mathbf{G}}(s) \in \mathcal{R}_z^{n\times m}(s)$, where $\mathcal{R}_x^{n\times m}(s)$ is the set of n by m rational proper transfer function matrices of 'order' x (its MacMillan degree). This problems is most commonly referred to as the model order reduction problem and has been the subject of considerable interest in recent years.

A key part of this problem is a measure of distance between two LTI systems. Algorithms proposed todate have utilized a variety of measures such as the $\mathcal{H}_2$-norm Valentin and Duc [1997] or the ν-Gap metric measure Cantoni [2001]. However, experience of robust control theory has shown that the most convenient and natural means of representing this distance is in terms of the induced $\mathcal{L}_2$ norm (the $\mathcal{H}_\infty$ norm) Skogestad and Postlethwaite [1996]. Indeed, the great majority of model order reduction techniques are based on the $\mathcal{H}_\infty$ norm and three of the most popular model reduction techniques, namely, the balanced truncation Moore [1981]Pernebo and Silverman [1982], balance residualization Fernando and Nicholson [1982], and the optimal Hankel-norm method Glover [1984] deal explicitly with $\left|\left|\mathbf{G}(s) - \hat{\mathbf{G}}(s)\right|\right|_\infty$. Thus, in the case of the balanced truncation and residualization it can be shown that the upper bound of the $\mathcal{H}_\infty$ norm is derived from the sum of the Hankel singular values of the deleted states Enns [1984]Glover [1984]Liu and Anderson [1989]. In the optimal Hankel norm reduction (HNR) approach, it can be shown that the rank of the Hankel operator $\mathcal{H}_G$ equals the order of the stable part of $\mathbf{G}(s)$ and the $\mathcal{L}_2$ gain of $\mathcal{H}_G$ never exceeds $||\mathbf{G}(s)||_\infty$ Glover [1984]Samar et al. [1995].

Unlike the LTI feedback optimization case where the most natural form of the problem formulation (for example as $\mathcal{H}_2$, or $\mathcal{H}_\infty$ norm minimization problem) has a satisfactory solution, no efficient solution is known for the most natural formulation of the model order reduction problem. Broadly speaking this is the consequence of the non-convexity of the set of all stable transfer functions of order less than a given value. Only the zero order case has been shown to have a convex formulation which makes it possible to find the global solution Grigoriadis and Skelton [1994b]Apkarian et al. [2003]Helmersson [1994]Xu and Lam [2003]. The consequence of this has been a verity of local and sub-optimal algorithms which solve some approximation of this problem, and where the aims are two fold; namely, to ensure that this approximation solution is as close as possible to the globally optimal solution, and the algorithm itself is reliable and well behaved.

The quintessential technique which has emerged within the LMI family of techniques is that of alternating projections. Several studies have shown that it is possible to employ this method to solve the model order reduction problem using LMIs Grigoriadis and Skelton [1994a]Grigoriadis [1997]Grigoriadis [1995b]. Alternating projections have been used successfully in the past in statistical estimation and image reconstruction problems Combettes [1993],Youla and Webb [1982]. The basic idea behind these techniques is that for a given family of convex sets, the sequence of alternating projections onto these sets converges to a point in the intersection of the family Grigoriadis [1995a]. However, for the case of non-convex sets - as encountered in model order reduction-convergence of the alternating projections is guaranteed only locally and hence when the initial starting point is in the vicinity of a feasible solution Grigoriadis and Skelton [1994b]Skelton et al. [1997]El Ghaoui and Niculescu [2000]. Inspired by alternating projections, and to circumnavigate the nonlinearly problem, almost all previously proposed LMI methods employ some sort of iteration Geromel et al. [2004]Helmersson [1994]Valentin and Duc [1997]Wortelboer et al. [1999] which can also include a rank minimization step Mesbahi and Papavassilopoulos [1997]Fazel et al. [2001] (itself is a non convex problem). As mentioned many of these algorithms require an initial solution and the great majority of these use the optimal Hankel norm method to obtain the initial feasible solution from which the process of iteration commences.

This paper proposes a new LMI-based method for model order reduction of LTI systems. Some of the notable benefits of the proposed method is that it involves no iterations, there are no requirements to provide an initial feasible solution, and there are no requirements for a priori choice of any slack variables.

2. BACKGROUND MATERIALS

2.1 Fundamentals of $\mathcal{H}_\infty$ LMI-based model order reduction

For the sake of definiteness, the type of systems considered in this paper are proper LTI systems $\mathbf{G}(\mathbf{s}) \in \mathcal{R}_q^{n\times m}(s)$ where $\mathcal{R}_q^{n\times m}(s)$ is the set described previously, and,

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D, \tag{1}$$

with the following state-space representation,

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ y &= Cx + Du,\end{aligned} \tag{2}$$

where $x \in \mathbf{R}^q$ are the states of the system, $y \in \mathbf{R}^n$ are the measured outputs, $u \in \mathbf{R}^m$ are the inputs and the dimensions of the state-space matrices are as follows, $A \in \mathbf{R}^{q\times q}$, $B \in \mathbf{R}^{q\times m}$, $C \in \mathbf{R}^{n\times q}$ and $D \in \mathbf{R}^{n\times m}$. Let $\hat{\mathbf{G}}(s) \in \mathcal{R}_z^{n\times m}(s)$ be an approximation to $\mathbf{G}(\mathbf{s})$ with order z. Let $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ be the state-space matrices of $\hat{\mathbf{G}}(s)$ with appropriate dimensions,

$$\hat{\mathbf{G}}(\mathbf{s}) \triangleq \left(\begin{array}{c|c} \hat{A} \in \mathbf{R}^{z\times z} & \hat{B} \in \mathbf{R}^{z\times m} \\ \hline \hat{C} \in \mathbf{R}^{n\times z} & \hat{D} \in \mathbf{R}^{n\times m}\end{array}\right). \tag{3}$$

The problem of finding $\hat{\mathbf{G}}(s)$, that is, the model order reduction problem, will then entail a minimization which in some form incorporates the 'distance' of $\mathbf{G}(s)$ and $\hat{\mathbf{G}}(s)$:

$$\begin{aligned} &\min\ z, \quad s.t. \\ &\left|\left|\mathbf{G}(\mathbf{s}) - \hat{\mathbf{G}}(\mathbf{s})\right|\right|_\infty < \beta. \end{aligned} \tag{4}$$

Formulation (**??**) has a natural LMI representation which can be achieved readily through the well known bounded real lemma Anderson and Vongpanitlerd [1973]. Specifically, let $\mathbf{H}(s) = \mathbf{G}(\mathbf{s}) - \hat{\mathbf{G}}(\mathbf{s})$, where,

$$\mathbf{H}(s) \triangleq \left(\begin{array}{c|c} \mathcal{A} \in \mathbf{R}^{(q+z)\times(q+z)} & \mathcal{B} \in \mathbf{R}^{(q+z)\times m} \\ \hline \mathcal{C} \in \mathbf{R}^{n\times(q+z)} & \mathcal{D} \in \mathbf{R}^{n\times m}\end{array}\right), \tag{5}$$

and define,

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \hat{A}\end{pmatrix}, \mathcal{B} = \begin{pmatrix} B \\ -\hat{B}\end{pmatrix}, \mathcal{C} = \begin{pmatrix} C & \hat{C}\end{pmatrix}, \mathcal{D} = \begin{pmatrix} D - \hat{D}\end{pmatrix}. \tag{6}$$

From the standard bounded real lemma $||\mathbf{H}(s)||_\infty < \gamma$ if there exists a positive definite, symmetric matrix $\bar{\mathcal{X}}$ that satisfies,

$$\begin{pmatrix} \underline{\mathcal{A}}^T\underline{\bar{\mathcal{X}}} + \underline{\bar{\mathcal{X}}\mathcal{A}} & \underline{\bar{\mathcal{X}}\mathcal{B}} & \underline{\mathcal{C}}^T \\ \underline{\mathcal{B}}^T\underline{\bar{\mathcal{X}}} & -\gamma I & \underline{\mathcal{D}}^T \\ \underline{\mathcal{C}} & \underline{\mathcal{D}} & -\underline{\gamma} I\end{pmatrix} < 0, \tag{7}$$

where all the matrix inequality variables have been underlined. Evidently (7) is a nonlinear matrix inequality, as it contains product of two variables. This is due to the fact that both the state-space matrices of $\hat{\mathbf{G}}(\mathbf{s})$ and the Lyapunov matrix $\bar{\mathcal{X}}$ have to be found. A routine way to deal with this is through the double sided projection lemma Gahinet and Apkarian [1992]. This gives a solvability condition of the original matrix inequality, only in terms of the state-space matrices of $\mathbf{G}(\mathbf{s})$ and $\bar{\mathcal{X}}$. This gives rise to the following equivalent problem,

$$\begin{pmatrix} A^T\mathcal{X} + \mathcal{X}A & B \\ B^T & -\gamma I\end{pmatrix} < 0, \tag{8}$$

$$\begin{pmatrix} A\mathcal{Y} + \mathcal{Y}A^T & C^T \\ C & -\gamma I\end{pmatrix} < 0, \tag{9}$$

$$\begin{pmatrix} \mathcal{Y} & I \\ I & \mathcal{X}\end{pmatrix} > 0. \tag{10}$$

where,

$$\bar{\mathcal{X}} = \begin{pmatrix} \mathcal{X} & \mathcal{N} \\ \mathcal{N}^T & \mathcal{Z}\end{pmatrix}, \tag{11}$$

$$\bar{\mathcal{Y}} = \begin{pmatrix} \mathcal{Y} & \mathcal{M} \\ \mathcal{M}^T & \mathcal{W}\end{pmatrix}. \tag{12}$$

Equations (8)-(10) are solvable LMIs. However, to see how the order constraint on $\hat{\mathbf{G}}(s)$ imposes a nonlinear constraint on (8)-(10), note that $\bar{\mathcal{Y}}$ is the inverse of $\bar{\mathcal{X}}$ and hence,

$$\begin{pmatrix} \mathcal{X} & \mathcal{N} \\ \mathcal{N}^T & \mathcal{Z}\end{pmatrix}\begin{pmatrix} \mathcal{Y} & \mathcal{M} \\ \mathcal{M}^T & \mathcal{W}\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}. \tag{13}$$

Therefore $\mathcal{X}\mathcal{Y} + \mathcal{N}\mathcal{M}^T = I$. Note that from (7), the dimensions of $\{\mathcal{X}, \mathcal{Y}\}$ and $\{\mathcal{Z}, \mathcal{W}\}$ are compatible with A and $\hat{A}$ respectively. Hence the dimensions of $\mathcal{N}, \mathcal{M}^T$ are $\in \mathbf{R}^{q\times(q-k)}$ and the product $\mathcal{N}\mathcal{M}^T$ has at least k eigenvalues which are zero, where $k = q - z$. Consequently, $\hat{\mathbf{G}}(\mathbf{s})$ has k states less than $\mathbf{G}(\mathbf{s})$ if and only if,

$$Rank(I - \mathcal{X}\mathcal{Y}) = q - k. \tag{14}$$

This becomes an additional constraint on LMIs (8)-(10) and the complete γ-suboptimal LMI problem of finding the best z-order approximation $\hat{\mathbf{G}}(s)$ to the q order original system $\mathbf{G}(s)$ becomes subjected to the solvability of the following problem with variables $(\mathcal{X}, \mathcal{Y}, \gamma)$,

$$\min\ \gamma \quad s.t.$$

$$\begin{pmatrix} A^T\mathcal{X} + \mathcal{X}A & B \\ B^T & -\gamma I\end{pmatrix} < 0, \tag{15}$$

$$\begin{pmatrix} A\mathcal{Y} + \mathcal{Y}A^T & C^T \\ C & -\gamma I\end{pmatrix} < 0, \tag{16}$$

$$\begin{pmatrix} \mathcal{Y} & I \\ I & \mathcal{X}\end{pmatrix} \geq 0, \tag{17}$$

$$Rank(I - \mathcal{X}\mathcal{Y}) = z. \tag{18}$$

where A, B, C are the state-space matrices of $\mathbf{G}(s)$, and $\mathcal{X}$ and $\mathcal{Y}$ are two positive definite symmetric matrices. The system described in (15)-(18) are only a feasibility problem. That is, if they are solvable, then it implies existence of a z-order LTI system $\hat{\mathbf{G}}(s)$ such that $\left|\left|\mathbf{G}(s) - \hat{\mathbf{G}}(s)\right|\right|_\infty < \gamma$. However, the actual solution $\hat{\mathbf{G}}(s)$ has to be recovered separately. Before proceeding to the general case of the z order system, we point out that (15)-(18) has two trivial solutions for which the problem is actually convex and standard computable. The first trivial case is for $z = q$ which means $\hat{\mathbf{G}}(s)$ is of equal order with $\mathbf{G}(s)$. In this case the rank constraint (18) disappears. The second trivial case for which the global solution can be computed, is $z = 0$, i.e. the zero order approximation to $\mathbf{G}(s)$. It is straightforward to show that for this case $\mathcal{A}_0 = A$, $\mathcal{B}_0 = B$ and $\mathcal{C}_0 = C$. Hence (7) becomes,

$$\begin{pmatrix} A^T\mathcal{X} + \mathcal{X}A & \mathcal{X}B & C^T \\ B^T\mathcal{X} & -\gamma I & D^T - \hat{D}^T \\ C & D - \hat{D} & -\gamma I\end{pmatrix} < 0, \tag{19}$$

where A, B, C are the state-space matrices of $\mathbf{G}(s)$, and $\mathcal{X}$ and $\mathcal{Y}$ are two positive definite symmetric matrices. The system described in (15)-(18) is only a feasibility problem. That is, if they are solvable, then it implies existence of a z-order LTI system $\hat{\mathbf{G}}(s)$ such that $\left|\left|\mathbf{G}(s) - \hat{\mathbf{G}}(s)\right|\right|_\infty < \gamma$. However, the actual solution $\hat{\mathbf{G}}(s)$ has to be recovered separately. In this paper, a simple factorization is used to propose a method of dealing with this rank constraint and obtain a locally optimal solution.

3. $\mathcal{H}_\infty$ LMI-BASED MODEL ORDER REDUCTION

In this section a new LMI method is proposed to solve the model order reduction problem. It is first shown that product of the two Lyapunov matrices $\mathcal{X}$ and $\mathcal{Y}$ in (15-18) is diagonalizable. Using this property new LMIs are proposed in terms of the eigenvalue-eigenvector factorization of the Lyapunov matrices, where the factorization is designed such that it is possible to specify how many eigenvalues in the product of the two matrices will be equal to 1. This will then cause $I-\mathcal{X}\mathcal{Y}$ to be rank deficient by the specified number and thus lead to a system with a prescribed order difference compared to the ordinal system.

Lemma 1. Let A and B be two symmetric positive definite matrices. Let $C = AB$. Then C is always diagonalizable.

Proof: Use standard singular value decomposition to show that,

$$A = U\Sigma U^T, \tag{20}$$

where the left and right singular vectors are the same because A is symmetric. Let $A = S^{-2}$. S may be calculated as follows,

$$S = A^{-1/2} = (U\Sigma U^T)^{-1/2} = U^{-T}\Sigma^{-1/2}U^{-1} \tag{21}$$

The matrix S is always computable since the singular vectors are invertible and given that $A > 0$, then $\Sigma > 0$ and hence $\Sigma^{-1/2}$ is also computable. Moreover S is also a symmetric positive definite matrix. Then,

$$C = AB = S^{-2}B \tag{22}$$

Using $\simeq$ to denote matrix similarity, it is now possible to show that,

$$C \simeq SCS^{-1} = SABS^{-1} = S^{-1}BS^{-1} \tag{23}$$

which is a symmetric matrix. Since C is similar to this matrix, then C must also be diagonalizable with real eigenvalues. $\Box$.

Corollary 2. Let A be a diagonalizable matrix with eigenvalues $\Lambda \in \mathbf{R}$. Then A may be expressed as the product of two symmetric matrices L and R, with real eigenvalues.

Proof: $A = W\Lambda W^{-1}$ where $\Lambda = Diag(\lambda_i)$. Let $\Lambda = \Lambda_1\Lambda_2$, then,

$$A = W\Lambda_1\Lambda_2 W^{-1} = (W\Lambda_1 W^T)(W^{-T}\Lambda_2 W^{-1}) = LR \tag{24}$$

since $\Lambda \in \mathbf{R}$ then it is always possible to find a pair of $\Lambda_1, \Lambda_2 \in \mathbf{R}$. Obviously L and R are symmetric positive definite matrices if and only if $\Lambda_1 > 0$ and $\Lambda_2 > 0$. Thus, if $A > 0$, then it is always possible to find $\Lambda_1 > 0$ and $\Lambda_2 > 0$. $\Box$.

Lemma 3. Let A and B be two square n-dimensional matrices. Then $Rank(I - AB) = k$ with $k < n$ if and only if $\lambda_i(AB) = 1$ with multiplicity of $n - k$.

Proof: $I - AB = I - W\Lambda W^{-1} = WW^{-1} - W\Lambda W^{-1} = W(I-\Lambda)W^{-1}$. Hence $\lambda_i(I - AB) = 1 - \lambda_i(AB)$ and for $n-k$ eigenvalues of $(I-AB)$ to be zero, then $\lambda_i(AB) = 1$ with multiplicity $n - k$. $\Box$

The main results of the paper can now be presented as follows.

Theorem 4. Let $\mathbf{G}(s)$ and $\hat{\mathbf{G}}(s)$ be as described in (1) and (3) with respective MacMillan degrees of q and z. Let $\hat{\mathcal{X}} \in \mathbf{R}^{q\times q}$ and $\hat{\mathcal{Y}} \in \mathbf{R}^{q\times q}$ be the globally optimal symmetric positive definite matrices which are obtained by solving (15)-(18) with $z = q$. Compute the standard eigenvalue decomposition,

$$\hat{\mathcal{X}}\hat{\mathcal{Y}} = \Pi\Lambda\Pi^{-1}, \tag{25}$$

where $\Lambda = Diag(\lambda_i)$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q$. Define $\Xi^T = \Pi^{-1}$ and let $\tilde{\mathcal{X}} \in \mathbf{R}^{z\times z}$ and $\tilde{\mathcal{Y}} \in \mathbf{R}^{z\times z}$ be two diagonal matrices of the specified dimensions. Let,

$$\Theta := \left\{\Theta \in \mathbf{R}^{q\times q}:\ \Theta > 0,\ \Theta = diag\{\theta_{ii}\}\right\}. \tag{26}$$

Define $I_0 \in \mathbf{R}^{q\times q} = Diag(0, I^{k\times k})$ with k being defined in (**??**), and $\tilde{\mathcal{X}}_0 \in \mathbf{R}^{q\times q}$ and $\tilde{\mathcal{Y}}_0 \in \mathbf{R}^{q\times q}$ as,

$$\tilde{\mathcal{X}}_0 = \begin{pmatrix} \tilde{\mathcal{X}} & 0 \\ 0 & 0 \end{pmatrix}, \tag{27}$$

$$\tilde{\mathcal{Y}}_0 = \begin{pmatrix} \tilde{\mathcal{Y}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{28}$$

Then there exists a sub-optimal approximation $\hat{\mathbf{G}}(s)$ to $\mathbf{G}(s)$ of order z, provided that the following systems of matrix inequalities in $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \Theta, \gamma)$ are solvable,

$$\min\ \gamma \qquad s.t. \qquad \begin{pmatrix} \tilde{\mathcal{Y}} & I \\ I & \tilde{\mathcal{X}} \end{pmatrix} > 0 \tag{29}$$

$$\begin{pmatrix} A^T\Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T + \Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T A & B \\ B^T & -\gamma I \end{pmatrix} < 0 \tag{30}$$

$$\begin{pmatrix} A\Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T + \Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T A^T & C^T \\ C & -\gamma I \end{pmatrix} < 0 \tag{31}$$

where A, B and C are the state-space matrices of $\mathbf{G}(s)$ as defined in (2).

Proof: Let,

$$\mathcal{X} = \Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T, \tag{32}$$

$$\mathcal{Y} = \Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T. \tag{33}$$

Then (30) is equivalent to (15) and (31) becomes (16). To see how the rank constraint (18) is satisfied, note that from corollary (2), we have,

$$\begin{aligned} I - \mathcal{X}\mathcal{Y} &= I - \Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T\Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T \\ &= I - \Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T \\ &= I - \Pi(\tilde{\mathcal{X}}_0\tilde{\mathcal{Y}}_0 + I_0)\Pi^{-1} \\ &= \Pi\begin{pmatrix} I - \tilde{\mathcal{X}}\tilde{\mathcal{Y}} & 0 \\ 0 & 0 \end{pmatrix}\Pi^{-1}. \end{aligned} \tag{34}$$

Consequently,

$$Rank(I - \mathcal{X}\mathcal{Y}) = Rank\begin{pmatrix} I - \tilde{\mathcal{X}}\tilde{\mathcal{Y}} & 0 \\ 0 & 0 \end{pmatrix} = z. \tag{35}$$

For the final part of the proof consider LMI (29) which is to ensure that (17) is positive definite. From (17) ,(32) and (33) it can be seen that,

$$\begin{aligned} \begin{pmatrix} \mathcal{Y} & I \\ I & \mathcal{X} \end{pmatrix} &= \begin{pmatrix} \Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T & I \\ I & \Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T \end{pmatrix} \\ &= \begin{pmatrix} \Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T & \Xi\Pi^T \\ \Pi\Xi^T & \Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T \end{pmatrix} \\ &= \Upsilon\begin{pmatrix} \tilde{\mathcal{Y}}_0 + I_0 & I \\ I & \tilde{\mathcal{X}}_0 + I_0\Theta \end{pmatrix}\Upsilon^T, \end{aligned} \tag{36}$$

where a few steps formulation will lead to the relationship,

$$\Upsilon = \begin{pmatrix} \Xi(I + I_0(\sqrt{\Theta^{-1}} - I)) & 0 \\ 0 & \Pi(I + I_0(\sqrt{\Theta} - I)) \end{pmatrix}. \tag{37}$$

Note that due to (26), then $\lambda_i(\sqrt{\Theta^{-1}} - I) \neq -1$ for $i = 1, \ldots, q$ which implies that Υ is never singular. Since for a nonsingular T the condition $TMT^T \geq 0$ is equivalent to $M \geq 0$, from (36), it follows that,

$$\begin{pmatrix} \mathcal{Y} & I \\ I & \mathcal{X} \end{pmatrix} \geq 0 \Rightarrow \begin{pmatrix} \tilde{\mathcal{Y}}_0 + I_0 & I \\ I & \tilde{\mathcal{X}}_0 + I_0\Theta \end{pmatrix} \geq 0$$
$$\Rightarrow \begin{pmatrix} \tilde{\mathcal{Y}} & 0 & I & 0 \\ 0 & I & 0 & I \\ I & 0 & \tilde{\mathcal{X}} & 0 \\ 0 & I & 0 & I \end{pmatrix} \geq 0, \quad (38)$$

from which it is straightforward to see that (38) has at least exactly $2k$ zero eigenvalues, and therefore a sufficient condition for (17) is (29). Consequently (15)-(18) are satisfied by (29)-(31) with condition (17) being explicitly satisfied. This completes the proof. □

Observe from (32) and (33) and corollary (2) that $\mathcal{X}$ and $\mathcal{Y}$ are symmetric for any choice of $\tilde{\mathcal{X}}$, $\tilde{\mathcal{Y}}$ and Θ. Furthermore, they are positive definite provided only $\tilde{\mathcal{X}} > 0$ and $\tilde{\mathcal{Y}} > 0$. However, matrix inequalities (29)-(31) are still not LMIs. This is due to the Θ^{-1} term which enters inequality (31) and makes it nonlinear. However, it is possible to convert this into a LMI by a simple change of variables and a transformation.

Theorem 5. Let A, B, C, Π, Ξ and $\tilde{\mathcal{X}}$ be as defined in Theorem 4, and,

$$J = \left(I^{k\times k}, 0^{k\times z}\right) \quad (39)$$

Denote $\breve{\mathcal{Y}}$ as a diagonal matrix of the same dimensions as $\tilde{\mathcal{Y}}$ and let,

$$\breve{\mathcal{Y}}_0 \in \mathbf{R}^{q\times q} = \begin{pmatrix} \breve{\mathcal{Y}} & 0 \\ 0 & 0 \end{pmatrix}. \quad (40)$$

Then, matrix inequalities (29)-(31) are equivalent to solving the following system of LMIs in $(\tilde{\mathcal{X}}, \breve{\mathcal{Y}}, \Theta, \gamma)$,

$$\min\ \gamma \quad s.t. \quad \begin{pmatrix} \breve{\mathcal{Y}} & J\Theta J^T \\ J\Theta J^T & \tilde{\mathcal{X}} \end{pmatrix} > 0, \quad (41)$$

$$\begin{pmatrix} A^T\Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T + \Pi(\tilde{\mathcal{X}}_0 + I_0\Theta)\Pi^T A & B \\ B^T & -\gamma I \end{pmatrix} < 0, \quad (42)$$

$$\begin{pmatrix} A\Xi(\breve{\mathcal{Y}}_0 + I_0\Theta)\Xi^T + \Xi(\breve{\mathcal{Y}}_0 + I_0\Theta)\Xi^T A^T & \Theta C^T \\ C\Theta & -\gamma I \end{pmatrix} < 0. \quad (43)$$

Proof: Define,

$$P = \begin{pmatrix} J\Theta J^T & 0 \\ 0 & I \end{pmatrix}, \quad (44)$$

and,

$$Q = \begin{pmatrix} \Theta & 0 \\ 0 & I \end{pmatrix}. \quad (45)$$

From (16) and (45) it can be seen that,

$$\begin{pmatrix} A\mathcal{Y} + \mathcal{Y}A^T & C^T \\ C & -\gamma I \end{pmatrix} < 0 \Rightarrow Q \begin{pmatrix} A\mathcal{Y} + \mathcal{Y}A^T & C^T \\ C & -\gamma I \end{pmatrix} Q^T < 0,$$
$$\Rightarrow \begin{pmatrix} \Theta(A\mathcal{Y} + \mathcal{Y}A^T)\Theta & \Theta C^T \\ C\Theta & -\gamma I \end{pmatrix} < 0. \quad (46)$$

Then using (33) yields,

$$\begin{aligned} &\Theta(A\mathcal{Y} + \mathcal{Y}A^T)\Theta \\ &\quad = \Theta(A\Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T + \Xi(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Xi^T A^T)\Theta \\ &\quad = A\Xi\Theta(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Theta\Xi^T + \Xi\Theta(\tilde{\mathcal{Y}}_0 + I_0\Theta^{-1})\Theta\Xi^T A^T \\ &\quad = A\Xi(\Theta^2\tilde{\mathcal{Y}}_0 + I_0\Theta)\Xi^T + \Xi(\Theta^2\tilde{\mathcal{Y}}_0 + I_0\Theta)\Xi^T A^T. \end{aligned} \quad (47)$$

and by (39) it can be shown that,

$$\Theta(I - I_0) = \begin{pmatrix} J\Theta J^T & 0 \\ 0 & 0 \end{pmatrix}. \quad (48)$$

It is then straightforward to show that,

$$\Theta^2\tilde{\mathcal{Y}}_0 = \begin{pmatrix} J\Theta^2 J^T\tilde{\mathcal{Y}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \breve{\mathcal{Y}} & 0 \\ 0 & 0 \end{pmatrix} = \breve{\mathcal{Y}}_0. \quad (49)$$

Substituting this in (47) and then inserting the results in (46) shows that (43) is equivalent to (31). Similarly, using (44) and (29), it can be shown that (29) is equivalent to (41) with the new matrix variable $\tilde{\mathcal{Y}} = (J\Theta J^T)^{-2}\breve{\mathcal{Y}}$. □

Note that matrix inequalities (41)-(43) are linear in their variables and thus are LMIs. In the next section a step-by-step procedure is given to use the results of this section to compute $\hat{\mathbf{G}}$.

4. EXAMPLES

In this section the proposed LMI model order reduction method is applied to several problems including a real-life model of a gas-turbine. The results are compared with the optimal Hankel-norm model order reduction which is widely accepted as the most reliable and effective model order reduction technique available. To this end, the performance of the resulting models is studied through a simple error functions defined by,

$$\mathcal{E}(\mathcal{G}_1, \mathcal{G}_2) = \mathcal{G}_1 - \mathcal{G}_2. \quad (50)$$

Accordingly $\mathcal{E}_L$ will be used to denote the error function which results from the LMI obtained results, and $\mathcal{E}_H$ is the respective optimal Hankel norm reduced model. Thus, the performance index of interest, referred to as 'the relative approximation error', will be defined as,

$$\Gamma = 100 \times \frac{||\mathcal{E}_H||_\infty - ||\mathcal{E}_L||_\infty}{||\mathbf{G}(s)||_\infty}. \quad (51)$$

Where $\mathbf{G}(s)$ is the system to be order reduced. Note that the magnitude of (51), $|\Gamma|$, indicates a large difference in the approximation error of the two techniques and the sign of Γ indicates which method performs better. A $\pm 1\%$ margin is defined on (51) within which the results of the two methods are referred to as being 'the same'. Therefore, for $\Gamma > 1\%$, the LMI method is considered to perform better than the optimal Hankel norm reduction (HNR) method. For $\Gamma < -1\%$, the HNR is said to give a better results, and for $|\Gamma| \leq 1$, the two techniques are said to produce the same results.

4.1 SISO example

In this example a simple SISO system is considered. The model is given by,

$$\mathbf{G}(s) = \frac{-0.737s^5 + 2s^4 - 0.756s^3 + 16.06s^2 - 0.632s + 27.48}{s^6 + 4.757s^5 + 23.4s^4 + 58.06s^3 + 101.9s^2 + 112.5s + 34.31} \quad (52)$$

This is a stable, but non-minimum phase system. For a brief treatment of model order reduction of non-minimum phase systems the reader can refer to Guo and Hwang [1996],Green [1988] and Bamani and Aldhaheri [1992]. A key feature of such systems is that when the number of RHP zeros is odd -(52) has three zeros- then the system will have inverse response characteristics. In such cases, it is desired that the reduced model contains the RHP zeros of the original system, or if this is not feasible (due to the sanctioned order difference), then to preserve some of the non-minimum phase properties. Let $\hat{\mathbf{G}}(k,s)$ denote the k^{th} order approximation of $\mathbf{G}(s)$. The problem to be solved is then,

$$\min \left|\left|\mathbf{G}(s) - \hat{\mathbf{G}}(k,s)\right|\right|_{\infty}. \tag{53}$$

Let $\hat{\mathbf{G}}_{LMI}(k,s)$ be the k^{th} reduced model order approximation obtained using the proposed LMI method and $\hat{\mathbf{G}}_{HNR}(k,s)$ be the k^{th} order optimal Hankel norm reduced model. Then,

$$\mathcal{E}_L = \mathcal{E}(\mathbf{G}, \hat{\mathbf{G}}_{LMI}), \tag{54}$$

$$\mathcal{E}_H = \mathcal{E}(\mathbf{G}, \hat{\mathbf{G}}_{HNR}). \tag{55}$$

The 1^{st} order approximation obtained using the LMI method was found to be,

$$\hat{\mathbf{G}}_{LMI}(1,s) = \frac{-0.09745s + 0.319}{s + 0.5055}, \tag{56}$$

and the corresponding HNR model obtained was,

$$\hat{\mathbf{G}}_{HNR}(1,s) = \frac{0.3711s + 1.092}{s + 1.22}. \tag{57}$$

Notice that the HNR model does not have a non-minimum phase zero, but the LMI model has one RHP zero. Additionally note that the poles of the models are considerably different. In particular, the poles of (52) are,

$$p(\mathbf{G}(s)) = \{-0.448, -1.638, -0.6681 \pm 1.941j, -0.667 \pm 3.263j,\} \tag{58}$$

where the dominant pole is at $s = -0.448$. Evidently, the pole of $\hat{\mathbf{G}}_{LMI}(1,s)$ which is at $s = -0.506$ is much closer to the dominant pole of $\mathbf{G}(s)$, than the pole of $\hat{\mathbf{G}}_{HNR}(1,s)$ which is at $s = -1.22$. This is reflected in the relative approximation error of the two models which is $\Gamma \approx 45.4\%$ and indicates that the approximation error of the LMI solution is almost half the HNR one. The responses of the models are shown in Figure 1.

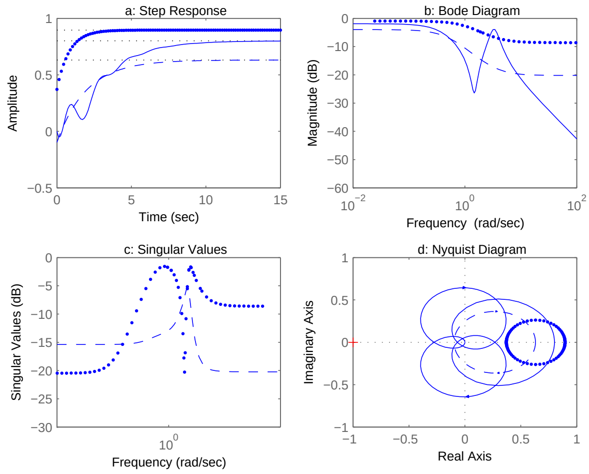

Fig. 1. SISO example results. Subplot (a) shows the step responses of $\mathbf{G}(s)$ (solid), $\hat{\mathbf{G}}_{LMI}(1,s)$ (dashed) and $\hat{\mathbf{G}}_{LMI}(1,s)$ (dotted). The Bode and Nyquist plots are shown in subplots (b) and (d) respectively and subplot (c) shows the singular values of the two error function $\mathcal{E}_L$ (dashed) and $\mathcal{E}_H$ (dotted).

4.2 Random multivariable systems

Let $\mathcal{G}_{\wp}(q,s)$ denote a random stable multivariable system with q-states. The formulations dealt with thus far in the paper concern the general non-square system. However, for the sake of compactness and without loss of generality, we consider square 10×10 dimensional systems in this example. A family of such randomly generated stable transfer functions are used in this section to benchmark the proposed LMI algorithm. Large system dimensions have been chosen in order to also test the stability of the algorithm. As such the following problem is to be solved,

$$\min \left|\left|\mathcal{G}_{\wp}(q,s) - \hat{\mathbf{G}}(k,s)\right|\right|_{\infty}, \tag{59}$$

where in (59) the s is the laplace operator, $\mathcal{G}_{\wp}(q,s) \in \mathcal{R}_q^{10\times10}(s)$ is a q-order randomly generated 10×10 transfer function matrix (TFM) and $\hat{\mathbf{G}}(k,s) \in \mathcal{R}_k^{10\times10}(s)$ is the k^{th} order approximation sought. Let,

$$\mathcal{E}_L = \mathcal{E}(\mathcal{G}_{\wp}(q,s), \hat{\mathbf{G}}_{LMI}(k,s)), \tag{60}$$

$$\mathcal{E}_H = \mathcal{E}(\mathcal{G}_{\wp}(q,s), \hat{\mathbf{G}}_{HNR}(k,s)), \tag{61}$$

where $\hat{\mathbf{G}}_{LMI}(k,s)$ and $\hat{\mathbf{G}}_{HNR}(k,s)$ are as defined previously and in (51), $\mathbf{G}(s) = \mathcal{G}_{\wp}(q,s)$. To make the test comprehensive, several different choices of (q,k) are considered and for every such choice, the problem is repeated for three different randomly generated systems. The Matlab function "`rss`" was utilized for this purpose. The specific range of the parameters considered in (59) are as follows,

$$q = \{10, 20, \ldots, 50\}, \tag{62}$$

$$k = \{1, 2, \ldots, 5\}. \tag{63}$$

The results of the resulting model order reduction problems are summarized in Figure 2. Points marked with $\square$ are cases where the LMI method performed worse than the HNR method, those marked with $\circ$ are cases where both methods has same performance, and points marked with $*$ are cases where the LMI performed better. The difference of performance in the results is indicated by the Z axis which shows the relative approximation error as described in Eq (51). In two cases, the LMI method was not able to produce good results, however in those cases the difference was not substantial. On the hand, for the majority of cases, the LMI method gives improvements over the HNR method.

REFERENCES

B.D.O. Anderson and S. Vongpanitlerd. *Network Analysis*. Prentice-Hall, Englewood Cliffs, NJ, 1973.

P. Apkarian, D. Noll, and H.D. Tuan. Fixed-order h_∞ control design via a partially augmented lagrangian method. *International Journal of Robust and Nonlinear Control*, 13:1137–1148, 2003.

A.H. Bamani and R.W. Aldhaheri. Combined schur/l_2-model reduction for nonminimal systems. *Optimal Control Application Methods*, 13:169–177, 1992.

A. J. Bosch. The factorization of a square matrix into two symmetric matrices. *American Mathematical Monthly*, 93:462–464, 1986.

A. J. Bosch. Note on the factorization of a square matrix into two hermitian or symmetric matrices. *SIAM Review*, 29(3):463–469, 1987.

Michael Cantoni. On model reduction in the ν-gap metric. *Proceedings of the 40th IEEE Conference on Decision and Control*, 2001. Orlando, Florida, USA.

P. Combettes. the foundations of set theoretic estimation. *Proc. IEEE*, 81:182–208, 1993.

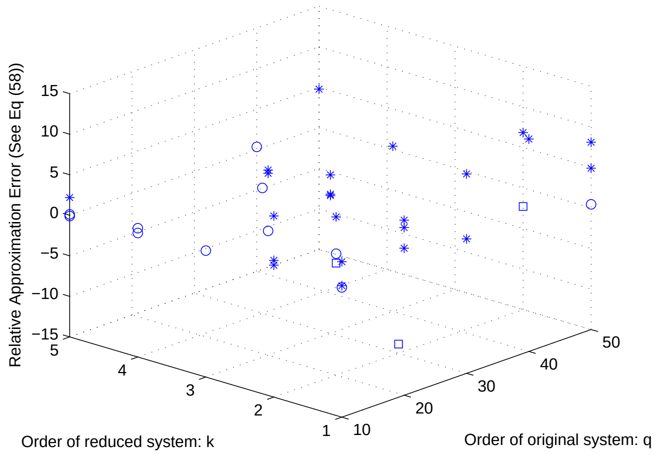

Fig. 2. Results for multivariable random tests. Points marked with □ are cases where the LMI method performed worse than the HNR method, those marked with O are cases where both methods has same performance, and points marked with $*$ are cases where the LMI performed better. The Z axis shows the relative approximation error.

L. El Ghaoui and S.I. Niculescu. *Advances in linear matrix inequality methods in control*, chapter 13. SIAM Philadelphia, 2000.

D. Enns. Model reduction with balanced realization: An error bound and a frequency weighted generalization. *Proceedings of the 23rd IEEE Conference on Decision and Control*, pages 127–32, 1984. Las Vegas, USA.

M. Fazel, H. Hindi, and S.P. Boyd. A rank minimization heuristic with application to minimum ordersystem approximation. *American Control Conference*, 2001. Arlington, VA, USA.

K.V. Fernando and H. Nicholson. Singular perturbational model reduction of balanced systems. *IEEE Transactions on Automatic Control*, 27(2):466–468, 1982.

G. Frobenius. Uber die mit einer matrix vertauschbaren matrizen. *Sitzungsber. Preuss. Akad. für Wiss*, pages 3–15, 1910.

P. Gahinet and P. Apkarian. A linear matrix inequality approach to h_∞ control. *International Journal of Control*, 4:421–448, 1992.

P. Gahinet and P. Apkarian. An lmi-based parameterization of all h_∞ controllers with applications. *Proceedings of the 32nd IEEE Conference on Decision and Control*, 1993. San Antonio, TX, USA.

Pascal Gahinet. Explicit controller formulas for lmi-based h_∞ synthesis. *Automatica*, 32(7):1007–1014, 1996.

J.C. Geromel, F.R.R. Kawaoka, and R.G. Egas. Model reduction of discrete time systems through linear matrix inequalities. *International Journal of Control*, 77(10): 978–984, 2004.

K. Glover. All optimal hankel-norm approximations of linear multivariable systems and their l_∞-error bounds. *International Journal of Control*, 39(6):1115–93, 1984.

M. Green. Balanced stochastic realization. *Linear Alegbra Applications*, 98:211–247, 1988.

K Grigoriadis and R. Skelton. Low-order control design for lmi problems using alternating projections. *Proceedings of the IEEE Conference on Decision and Control*, 1994a. Orlando, USA.

K Grigoriadis and R. Skelton. Alternating convex projection methods for covariance control design. *International Journal of Control*, 60:1083–1106, 1994b.

K.M. Grigoriadis. l_2 and l_∞ model reduction via linear matrix inequalities. *International Journal of Control*, 68:485–498, 1997.

K.M. Grigoriadis. Optimal h_∞ model reduction via linear matrix inequalities: Continuous and discrete-time cases. *Proceedings of the 34th IEEE Conference on Decision and Control*, 1995a. New Orleans, USA.

K.M. Grigoriadis. Optimal h_∞ model reduction via linear matrix inequalities: continuous and discrete-time cases. *Systems and Control Letters*, 26:321–333, 1995b.

T.Y. Guo and C. Hwang. Optimal reduced-order models for unstable and nonminimum-phase systems. *IEEE Transactions on Circuits and Systems-I: Fundumental Theory and Applications*, 43(9):800–805, 1996.

A. Helmersson. Model reduction using lmis. *Proceedings of the 33th IEEE Conference on Decision and Control*, 1994. Lake Buena Vista, USA.

Y. Liu and B.D.O. Anderson. Singular perturbation approximation of balanced systems. *International Journal of Control*, 50(4):1379–1405, 1989.

M. Mesbahi and G.P. Papavassilopoulos. On the rank minimization problem over a positive semidefinite linear matrix inequality. *IEEE Transactions on Automatic Control*, 42(2):239–243, 1997.

M. Moore. Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Transactions on Automatic Control*, 26(1):17–32, 1981.

L. Pernebo and L.M. Silverman. Model reduction by balanced state space representation. *IEEE Transactions on Automatic Control*, 27(2):382–387, 1982.

R. Samar, I. Postlethwaite, and D.W. Gu. Model reduction with balanced realizations. *International Journal of Control*, 62(1):33–64, 1995.

C.W. Scherer, P. Gahinet, and M. Chillali. Multiobjective output feedback control via lmi optimization. *IEEE Transc. Automatic Control*, 42:896–911, 1997.

R.E. Skelton, K.M. Girgoriadis, and T. Iwasaki. Control system design: A unified algebraic approach. *American Control Conference, Workshop No. 5*, 1997. Albuquerque, USA.

S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design*. Wiley, 1996.

H.D. Tuan, P. Apkarian, and T.Q. Nguyen. Robust and reduced-order filtering: New lmi-based characterization and methods. *IEEE Transactions on signal processing*, 49(12):2975–2984, 2001.

C. Valentin and G. Duc. Lmi-based algorithms for frequency weighted optimal h_2-norm model reduction. *Proceedings of the 36th IEEE Conference on Decision and Control*, 1997. San Diego, USA.

P.M.R. Wortelboer, M. Steinbuch, and O.H. Bosgra. Iterative model and controller reduction using closed-loop balancing, with application to a compact disc mechanism. *International Journal of Robust and Nonlinear Control*, 9(3):123–142, 1999.

Shengyuan Xu and James Lam. h_∞ model reduction for discrete-time singular systems. *Systems and Control Letters*, 48:121–133, 2003.

D. Youla and H. Webb. Image reconstruction by the method of convex projections: Part 1-theory. *IEEE Trans. Medical Imag.*, 1:81–94, 1982.