Output Feedback Sampled-Data Control of Nonlinear Systems in Output Feedback Form

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Abstract: This paper deals with sampled-data control of nonlinear systems in the output feedback form. A sampled-data control strategy is proposed based on the existing control design in the continuous-time domain via output feedback. The proposed control uses the sampled output and a discrete-time implementation of filters involved. The overall stability of the system under the proposed control has been analyzed, and the semi-global asymptotic stability for the system with relative degree one and two is established by keeping the sampling interval with a specified range.

Keywords: sampled-data control, nonlinear systems, output feedback form

1. INTRODUCTION

Digital computers have been widely used for implementation of control strategies. For linear systems, computer implementation can be carried out based on the discrete-time version of the control design, as there is always discrete time version which has the same structure as the original continuous-time model. Sampled-data control of linear systems has been intensively studied in literature, see Chen and Francis [1995] for instance.

However most of the engineering systems are nonlinear, and there have been tremendous progresses in control design for nonlinear systems in the last decade, and a number of control design methods and control strategies have been proposed. To implement the nonlinear control laws for nonlinear systems in discrete time requires investigating the performance of nonlinear sampled-data systems. Unfortunately the results on sampled-data control of linear systems can not be applied directly to nonlinear sampled-data systems, due to the fact that unlike in the linear context, an exact, discrete-time model of a general nonlinear system is hard to obtain and then not available for controller design. Furthermore, most of the continuous-time nonlinear theories are not applicable to the sampled-data case, since they rely on particular structures of nonlinear systems, which are usually destroyed by sampling, for example, feedback linearizability [Grizzle and Kokotovic, 1988]. These facts strongly motivate the research on nonlinear sampled-data control systems, to which a great deal of attention has been drawn recently, see Hou et al. [1997], Teel et al. [1998], Dabroom and Khalil [2001], Khalil [2004] and Nesic and Teel [2004].

Results on the problem of output feedback sampled-data control of nonlinear systems have been reported in Dabroom and Khalil [2001] and Khalil [2004]. Both of Dabroom and Khalil [2001] and Khalil [2004] employed the emulation method to design sampled-data controllers for the same class of systems, while continuous-time controllers are actually state-feedback controllers using high-gain observers. The difference is that in Dabroom and Khalil [2001] the high-gain observer was designed in continuous time while in Khalil [2004], on discrete-time models. The results in Dabroom and Khalil [2001] and Khalil [2004] showed that, for some class of nonlinear systems, the obtained sampled-data controller can recover the performance of the continuous-time state feedback controller, provided that the sampling period $T \to 0$.

In this paper, we will concentrate on sampled-data control of nonlinear systems in the output feedback form. We will investigate the conditions, particularly those on the sampling period T, under which the sampled-data controller will stabilize the close-loop sampled-data system. The paper makes full use of existing methods for designing continuous-time controllers for the nonlinear systems in the output feedback form Marino and Tomei [1995] and Ding [2003].

2. PROBLEM STATEMENT

We consider sampled-data control of one class of single-input-single-output nonlinear systems which can be transformed into the following output feedback form

$$\begin{aligned} \dot{x} &= A_c x + \phi(y) + bu \\ y &= Cx \end{aligned} \tag{1}$$

with

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix}$$

$$C = [1 \quad 0 \quad \cdots \quad 0]$$

where $x \in R^n$ is the state vector, u, $y \in R$ the input and the output respectively, $b \in R^n$ an known vector, while $\phi(y)$ is a smooth nonlinear vector field satisfying $\phi(0) = 0$.

Assumption 1. The system is of minimum phase, i.e., the polynomial $\mathcal{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ is Hurwitz.

The problem considered here can be described as follows: For system (1), if there exists a continuous-time output feedback controller u_c given by

$$\dot{\omega}(t) = \nu(y, \omega), \quad u_c = u_c(y, \omega) \tag{2}$$

which renders the origin of system (1) asymptotically stable, then for any given initial set, there exists a constant $T^* > 0$ such that for all $0 < T < T^*$, the system will still be asymptotically stabilized by the sampled-data controller u_d

$$\begin{aligned} \omega(m) &= \nu_d\big(y(m-1), \omega(m-1)\big) \\ u_d(t) &= u_c(y(m), \omega(m)), \quad \forall t \in [mT, mT+T) \end{aligned} \tag{3}$$

where ν_d is some discrete-time implementation of the system ω, T is the sampling period and $\omega(n)$ and $y(n)$ denote the signals ω and y at the nth sampling point respectively, that is, $\omega(m) = \omega(mT)$, $y(m) = y(mT)$

3. CONTINUOUS-TIME CONTROL DESIGN

In this paper we focus our discussion on the cases of relative degree one and two.

3.1 State transformation

For system (1) with relative degree $\rho = 2$, we can introduce the following filter [Marino and Tomei, 1995]

$$\dot{\xi} = -\lambda\xi + u \tag{4}$$

where $\lambda > 0$ is the design parameter. With the vector $\bar{d} = b \in R^n$, the following filtered transformation $\eta = x - \bar{d}\xi$ can transform system (1) into

$$\dot{\eta} = A_c\eta + \phi(y) + d\xi, \quad y = C\eta \tag{5}$$

where $d = [A_c\bar{d} + \lambda\bar{d}]$. It can be shown that $d_1 = b_\rho$ and

$$\sum_{i=1}^{n} d_i s^{n-i} = \mathcal{B}(s)(s + \lambda)$$

Since λ is positive, d is a Hurwitz vector with $d_1 = b_\rho = 1$ (here we assume $b_\rho = 1$ without loss of generality). Therefore with ξ being the input, system (5) is of minimum phase and relative degree one. To extract the internal dynamics of (5), introduced is the following state transform

$$\begin{aligned} z_1 &= \eta_2 - d_2\eta_1 \\ &\vdots \\ z_{n-1} &= \eta_n - d_n\eta_1 \\ y &= \eta_1 \end{aligned} \tag{6}$$

where $z \in R^{n-1}$. In the new coordinates, system (5) can be written as

$$\begin{aligned} \dot{z} &= Dz + \phi_z(y) \\ \dot{y} &= z_1 + \phi_y(y) + \xi \end{aligned} \tag{7}$$

where D is the companion matrix of d and is given by

$$D = \begin{bmatrix} -d_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_n & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\phi_z(y) = y \begin{bmatrix} d_3 - d_2^2 \\ d_4 - d_3 d_2 \\ \vdots \\ d_n - d_{n-1}d_2 \\ -d_n d_2 \end{bmatrix} + \begin{bmatrix} \phi_2 - \phi_1 d_2 \\ \phi_3 - \phi_1 d_3 \\ \vdots \\ \phi_{n-1} - \phi_1 d_{n-1} \\ \phi_n - \phi_1 d_n \end{bmatrix}$$

$$\phi_y(y) = \phi_1 + y d_2$$

where ϕ_i denotes the ith component of the vector ϕ. Finally the model for control design is the extended system consisting of (7) and (4).

3.2 Control for the case $\rho = 1$

If system (1) is of relative degree one, then we are actually dealing with the following system

$$\begin{aligned} \dot{z} &= Dz + \phi_z(y) \\ \dot{y} &= z_1 + \phi_y(y) + u_{c1} \end{aligned} \tag{8}$$

and the continuous-time control u_{c1} is designed as

$$u_{c1} = -\phi_y - k_1 y - y^{-1}\phi_z^{\mathrm{T}} P^2 \phi_z \tag{9}$$

where k is a positive real and P is the symmetric positive definite solution of the Lyapunov equation

$$D^{\mathrm{T}}P + PD = -(\gamma_1 + 2)I \tag{10}$$

with γ_1 being a positive real.

3.3 Control for the case $\rho = 2$

In the case of relative degree $\rho = 2$, backstepping technique will be employed to find the final control u_{c2} from the desirable value of ξ. Indeed, we have the following control

$$u_{c2} = -y + \lambda\hat{\xi} + \frac{\partial\hat{\xi}}{\partial y}(\phi_y + \xi) - \tilde{\xi}\left(\frac{\partial\hat{\xi}}{\partial y}\right)^2 \tag{11}$$

where $\hat{\xi} = -\phi_y - k_1 y - y^{-1}\phi_z^{\mathrm{T}}P^2\phi_z$, and $\tilde{\xi} = \xi - \hat{\xi}$, which is governed by

$$\dot{\tilde{\xi}} = -\lambda\xi - \frac{\partial\hat{\xi}}{\partial y}\dot{y} + u \tag{12}$$

It is easy to verify that the stabilizing function $\hat{\xi}$ is a function of y and $\hat{\xi}(0) = 0$, while the continuous-time control u_{c2} is a function of y and ξ and $u_{c2}(0,0) = 0$.

3.4 Stability of the continuous-time system

If $\rho = 2$, the time derivative of the following Lyapunov function

$$V_{c2} = z^{\mathrm{T}}Pz + \frac{1}{2}y^2 + \frac{1}{2}\tilde{\xi}^2 \tag{13}$$

satisfies

$$
\begin{aligned}
\dot V_{c2} &= -(\gamma_1+2)\|z\|^2 + 2z^{\mathrm T}P\phi_z \\
&\quad + y(z_1 - k_1 y - y^{-1}\phi_z^{\mathrm T}P^2\phi_z + \tilde\xi) \\
&\quad + \tilde\xi\Big(-\lambda\tilde\xi - y - \tilde\xi\frac{\partial\hat\xi}{\partial y}z_1 - \tilde\xi\Big(\frac{\partial\hat\xi}{\partial y}\Big)^2\Big) \\
&\le -k_1y^2 - \gamma_1\|z\|^2 - \lambda\tilde\xi^2 - (\|z\| - \|P\phi_z\|)^2 \\
&\quad -\frac{3}{4}\|z\|^2 - \Big(\frac12 z_1 + \tilde\xi\frac{\partial\hat\xi}{\partial y}\Big)^2 \\
&\le -k_1y^2 - \gamma_1\|z\|^2 - \lambda\tilde\xi^2
\end{aligned} \tag{14}
$$

which proves by the theorem 4.10 in Khalil [2002] that $(y, z, \tilde\xi) = 0$ is an exponentially stable equilibrium point of the extended system of (7) and (4), which implies that $(y, z, \xi) = 0$ is globally asymptotically stable and so is the origin $(x, \xi) = 0$. In case $\rho = 1$, we take $V_{c1} = z^{\mathrm T}Pz + \frac12 y^2$ as the Lyapunov function, and the same conclusion can be reached.

4. STABILITY ANALYSIS

The following lemma is needed for stability analysis in sampled-data case.

Lemma 2. Let $V : R^n \to R$ be a continuously differentiable, radically unbounded, positive definite function. Define $\mathcal{D} := \{\chi \in R^n | V(\chi) \le r\}$ with $r > 0$. Suppose

$$\dot V \le -\alpha V + \beta V_m, \quad \forall t \in (mT, (m+1)T], \tag{15}$$

hold for all $\chi(mT) \in \mathcal{D}$, where α, β are any given positive reals with $\alpha > \beta$, $T > 0$ the fixed sampling period and $V_m := V(\chi(mT))$. If $\chi(0) \in \mathcal{D}$, then the following holds:

$$\lim_{t\to\infty} \chi(t) = 0 \tag{16}$$

Proof. Since $\chi(0) \in \mathcal{D}$, then (15) holds for $t \in (0, T]$ with the following form

$$\dot V \le -\alpha V + \beta V(\chi(0)).$$

Using comparison lemma [Khalil, 2002] it is easy to get from the above that for $t \in (0, T]$,

$$
\begin{aligned}
V(\chi(t)) &\le e^{-\alpha t}V_0 + \frac{1 - e^{-\alpha t}}{\alpha}\beta V_0 \\
&= q(t)V_0,
\end{aligned} \tag{17}
$$

where

$$q(t) := \Big(e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t})\Big)$$

Since $\alpha > \beta > 0$, then $q(t) \in (0, 1)$, $\forall t \in (0, T]$. Then we have

$$V(\chi(t)) < V_0, \ \forall t \in (0, T]. \tag{18}$$

Particularly, letting $t = T$ in (17) leads to $V_1 \le q(T)V_0$, which means that $V_1 \in \mathcal{D}$. Therefore (15) holds for $t \in (T, 2T]$. By induction, we have

$$V(\chi(t)) < V_m, \ \forall t \in (mT, (m+1)T]. \tag{19}$$

which states intersample behaviour of the sampled-data system concerned, and in particular

$$V_{m+1} \le q(T)V_m \tag{20}$$

indicating that V decreases at two consecutive sampling points with a fixed ratio. From (20),

$$V_m \le q(T)V_{m-1} \le q^m(T)V_0 \tag{21}$$

which implies that $\lim_{m\to\infty} V_m = 0$. The conclusion then follows from (19), which completes the proof.

4.1 Stability of the sampled-data system with $\rho = 1$

In this case, the sampled-data system takes the following form

$$
\begin{aligned}
\dot z &= Dz + \phi_z(y) \\
\dot y &= z_1 + \phi_y(y) + u_{d1}
\end{aligned} \tag{22}
$$

where u_{d1} is the sampled-data controller and can be simply implemented by

$$u_{d1}(t) = u_{c1}(y(n)) \tag{23}$$

for all $t \in [nT, nT + T)$.

For the stability of the closed-loop sampled-data system, we have the following result.

Theorem 3. For system (22) with the sampled-data controller u_{d1} shown in (23) and for any given initial set containing the origin, there exists a constant $T_1 > 0$ such that, for all $0 < T < T_1$, the overall system is asymptotically stable.

Proof. Define $\chi := [z, y]^{\mathrm T}$. Let $B_r := \{\chi \in R^n | \ ||\chi|| \le r\}$ with r any given positive real. We still choose $V_{d1}(\chi) = z^TPz + \frac12 y^2$ as the Lyapunov function candidate for the sampled-data system. Then we present some sets used throughout the proof. Define $c := \max_{\chi\in B_r} V_{d1}(\chi)$ and the set $\Omega_c := \{\chi \in R^n | V_{d1}(\chi) \le c\}$. There exist two $\mathcal{K}$ functions ψ_1 and ψ_2 such that $\psi_1(||\chi||) \le V_{d1}(\chi) \le \psi_2(||\chi||)$. Let $l := \psi_1^{-1}(c) + \nu$ with ν an arbitrarily positive number. Define $B_l := \{\chi \in R^n | \ ||\chi|| \le l\}$. Then $B_r \subset \Omega_c \subset B_l$. The constants L_{u1}, L_1, L_2 are Lipschitz constants of u_{c1}, ϕ_y and ϕ_z, respectively, chosen with respect to B_l.

These local Lipschitz conditions establishes that for the overall sampled-data system with $\chi(0) \in \Omega_c$, there exists a unique solution $\chi(t)$ over some interval $[0, t_1)$. Notice that t_1 might be finite. However, later analysis shows that the solution does exist for every sampling interval, and thus exists for all $t \ge 0$ with the property that $\lim_{t\to\infty} \chi(t) = 0$.

Consider the case when $t = 0$, $\chi(0) \in B_r \subset \Omega_c$. l can be chosen sufficiently large such that there exists a $T_1* \in (0, t_1)$, and for all $T \in (0, T_1^*)$, the following holds:

$$\chi(t) \in B_l, \ \forall t \in [0, T], \chi(0) \in \Omega_c \tag{24}$$

The existence of T_1^* is ensured by continuous dependency of the solution $\chi(t)$ on the initial conditions.

Next we shall estimate $|y(t) - y(0)|$ during the interval [0,T] forced by the sampled-data control u_{d1}. From the second equation of (22), the dynamics of y is

$$\dot y = z_1 + \phi_y(y) + u_{d1}$$

It then follows that

$$y(t) = y(0) + \int_0^t z_1(\tau)d\tau + \int_0^t u_{d1}(\tau)d\tau + \int_0^t \Big(\phi_y(y) - \phi_y\big(y(0)\big)\Big)d\tau + \int_0^t \phi_y\big(y(0)\big)d\tau \quad (25)$$

Then we have

$$|y(t) - y(0)| \le \underbrace{\int_0^t \|z(\tau)\|d\tau}_{\Delta_1} + \int_0^t L_{u1}|y(0)|d\tau + \int_0^t L_1|y(\tau) - y(0)|d\tau + \int_0^t L_1|y(0)|d\tau \quad (26)$$

We first calculate the integral Δ_1. From the first equation of system (7), we obtain

$$z(t) = e^{Dt}z(0) + \int_0^t e^{Dt}\phi_z\big(y(\tau)\big)d\tau \quad (27)$$

Since D is a Hurwitz matrix, there exist positive reals k_2, σ such that $\|e^{Dt}\| \le k_2 e^{-\sigma t}$. Thus,

$$\begin{aligned}\|z(t)\| \le{}& k_2 e^{-\sigma t}\|z(0)\| \\ &+ \int_0^t k_2 e^{-\sigma(t-\tau)}\|\phi_z\big(y(\tau)\big) - \phi_z\big(y(0)\big)\|d\tau \\ &+ \int_0^t k_2 e^{-\sigma(t-\tau)}\|\phi_z\big(y(0)\big)\|d\tau \\ \le{}& k_2 e^{-\sigma t}\|z(0)\| \\ &+ L_2\int_0^t k_2 e^{-\sigma(t-\tau)}|y(\tau) - y(0)|d\tau \\ &+ L_2\int_0^t k_2 e^{-\sigma(t-\tau)}|y(0)|d\tau \end{aligned} \quad (28)$$

Then the following inequality holds

$$\Delta_1 \le \frac{k_2\|z(0)\|}{\sigma}\big(1 - e^{-\sigma t}\big) + \frac{k_2 L_2}{\sigma}|y(0)|t + \frac{k_2 L_2}{\sigma}\int_0^t |y(\tau) - y(0)|d\tau \quad (29)$$

Now we are ready to compute $|y(t) - y(0)|$. In fact, from (26) and (29), we can see

$$|y(t) - y(0)| \le A_1\big(1 - e^{-\sigma t}\big) + B_1 t + H\int_0^t |y(\tau) - y(0)|d\tau \quad (30)$$

where $A_1 = \sigma^{-1}k_2\|z(n)\|$, $B_1 = L_{u1}|y(0) + L_1|y(0)| + \sigma^{-1}k_2 L_2|y(0)|$ and $H = \sigma^{-1}k_2 L_2 + L_1$. Applying Gronwall-Bellman inequality Khalil [2002] to (30) produces

$$\begin{aligned}|y(t) - y(0)| \le{}& A_1\big(1 - e^{-\sigma t}\big) + \frac{B_1}{H}\big(e^{Ht} - 1\big) \\ &+ A_1\big(\sigma e^{Ht} + He^{-\sigma t} \\ &- (H + \sigma)\big)(H + \sigma)^{-1}\end{aligned} \quad (31)$$

Setting $t = T$ in the right side of (31) leads to

$$|y(t) - y(0)| \le \delta_1(T)|y(0)| + \delta_2(T)||z(0)|| \quad (32)$$

where

$$\begin{aligned}\delta_1(T) ={}& H^{-1}(L_{u1} + L_1 + \sigma^{-1}k_2 L_2)\big(e^{HT} - 1\big) \\ \delta_2(T) ={}& \sigma^{-1}k_2\big(\sigma e^{HT} + He^{-\sigma T} - (H + \sigma)\big)(H + \sigma)^{-1} \\ &+ \sigma^{-1}k_2\big(1 - e^{-\sigma T}\big)\end{aligned} \quad (33)$$

Next we shall study the behaviour of the sampled-data system using the chosen Lyapunov function candidate V_{d1}. When $t \in (0, T)$, its time derivative satisfies

$$\begin{aligned}\dot{V}_{d1} ={}& -(\gamma_1 + 2)\|z\|^2 + 2z^{\mathrm{T}}P\phi_z + y(z_1 + \phi_y + u_{d1}) \\ \le{}& -k_1 y^2 - \gamma_1||z||^2 + |y||u_{d1} - u_{c1}| \\ \le{}& -k_1 y^2 - \gamma_1||z||^2 + L_{u1}|y - y(0)|y| \\ \le{}& -\big(k_1 - \frac{L_{u1}}{2}(\delta_1(T) + \delta_2(T))\big)y^2 - \gamma_1||z||^2 \\ &+ \frac{L_{u1}}{2}\delta_1(T)|y(0)|^2 + \frac{L_{u1}}{2}\delta_1(T)||z(0)||^2 \\ \le{}& -\alpha_1(T)V_{d1} + \beta_1(T)V_{d1}(\chi(0))\end{aligned} \quad (34)$$

where

$$\begin{aligned}\alpha_1(T) ={}& \min\left\{2k_1 - L_{u1}\delta_1(T) - L_{u1}\delta_2(T), \frac{\gamma_1}{\lambda_{max}(P)}\right\} \\ \beta_1(T) ={}& \max\left\{L_{u1}\delta_1(T), \frac{L_{u1}\delta_2(T)}{2\lambda_{min}(P)}\right\}\end{aligned} \quad (35)$$

where $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote the maximum and minimum eigenvalues of a matrix, respectively.

It then can be shown that there exists a $T_2^* > 0$ so that $\alpha_1(T) > \beta_1(T) > 0$, for all $T \in (0, T_2^*)$. Note from (33) that both $\delta_1(T)$ and $\delta_2(T)$ are actually the continuous functions of T with $\delta_1(0) = \delta_2(0)$, which implies $\beta_1(0) = 0$. It can then be established from (35) that the function $e_1(T) := \alpha_1 - \beta_1$ is a decreasingly continuous function of T with $e_1(0) > 0$, which asserts by the continuity of $e_1(T)$ the existence of T_2^*.

Finally, set $T_1 = \min(T_1^*, T_2^*)$. From Lemma 2 we have $V_1 \in \Omega_c$. Thus the above analysis can be repeated for next interval $[T, 2T)$, and for each interval $[mT, mT + T)$. Then the conclusion follows from Lemma 2.

4.2 Stability of the sampled-data system with $\rho = 2$

We first give out the implementation of the sampled-data controller u_{d2}. Indeed u_{d2} is given as follows

$$u_{d2}(t) = u_{c2}\big(y(m), \xi(m)\big), \qquad \forall t \in [mT, mT + T) \quad (36)$$

where $y(m)$ can be obtained by sampling $y(t)$ at each sampling instant, while $\xi(m)$, as one part of the controller, is obtained digitally by

$$\xi(m) = e^{\lambda T}\xi(m-1) + \lambda^{-1}(1-e^{-\lambda T})u_{d2}\big(y(m-1),\xi(m-1)\big) \quad (37)$$

Then we have the following theorem:

Theorem 4. For the system of (7) and (4) with the sampled-data controller shown in (36) and (37), and for any given initial set containing the origin, there exists a constant $T_2 > 0$ such that for all $0 < T < T_2$, the controller u_{d2} asymptotically stabilizes the system.

Proof. In this case the analysis is more involved than that for the case $\rho = 1$, as the effect of the dynamic filter of ξ has to be dealt with.

Notice from (37) that $\xi(n)$ is the exact, discrete-time model of the system

$$\dot{\xi} = -\lambda\xi + u_{d2}, \quad (38)$$

since u_{d2} remains constant during each interval and the dynamics of ξ shown in (38) is linear. This indicates that we can use (38) instead of (37) for the stability analysis of the sampled-data system, as (37) and (38) are virtually equivalent at each sampling instant.

Next define $\chi = [z, y, \tilde{\xi}]^T$. Then a Lyapunov function candidate is chosen as the same as for continuous-time case, that is, $V_{d2}(\chi) = z^{\mathrm{T}}Pz + \frac{1}{2}y^2 + \frac{1}{2}\tilde{\xi}^2$. Similar to the case of $\rho = 1$, the sets B_r, Ω_c and B_l can also be defined such that $B_r \subset \Omega_c \subset B_l$, and there exists a $T_3^* > 0$ such that for all $T \in (0, T_3^*)$, the following holds

$$\chi(t) \in B_l,\ \forall t \in (0,T], \chi(0) \in \Omega_c. \quad (39)$$

Consider the case when $t = 0$, $\chi(0) \in B_r \subset \Omega_c$. Then we shall compute the estimates of $|\xi(t)-\xi(0)|$ and $|y(t)-y(0)|$ for $t \in (0,T)$. From (38), we have

$$\xi(t) = e^{\lambda t}\xi(n) + \int_0^t e^{\lambda(t-\tau)}u_{d2}d\tau \quad (40)$$

Then, using the Lipschitz property of u_{c2} and the fact that $u_{d2}(0,0) = 0$, it can be shown that

$$\begin{aligned}|\xi(t)-\xi(0)| &\le (1-e^{-\lambda t})|\xi(0)| \\ &\quad + |u_d\big(y(0),\xi(0)\big)| \int_0^t e^{-\lambda(t-\tau)}d\tau \\ &\le \delta_3(T)|y(0)| + \delta_4(T)|\xi(0)| \end{aligned}\quad (41)$$

where $\delta_3(T) = \lambda^{-1}L_{u2}\big(1-e^{-\lambda T}\big)$, $\delta_4(T) = (\lambda^{-1}L_{u2}+1)\big(1-e^{-\lambda T}\big)$ and L_{u2} is a Lipschitz constant of u_{c2} with respect to the set B_l. As for $|y-y(0)|$, we have from (7)

$$\begin{aligned} y(t) &= y0) + \int_0^t z_1(\tau)d\tau + \int_0^t \xi(\tau)d\tau \\ &\quad + \int_0^t \Big(\phi_y(y) - \phi_y\big(y(0)\big)\Big)d\tau \\ &\quad + \int_0^t \phi_y\big(y(0)\big)d\tau \end{aligned} \quad (42)$$

It can then be shown that

$$\begin{aligned}|y(t)-y(0)| &\le \underbrace{\int_0^t \|z(\tau)\|d\tau}_{\Delta_1} + \underbrace{\int_0^t \|\xi(\tau)\|d\tau}_{\Delta_2} + \\ &\quad \int_0^t L_1|y(\tau)-y(0)|d\tau + \int_0^t L_1|y(0)|d\tau \end{aligned}\quad (43)$$

where Δ_1 is already shown in (29). It can be obtained from (40) that

$$\Delta_2 \le \frac{|\xi(0)|}{\lambda}\big(1-e^{-\lambda t}\big) + \frac{L_u}{\lambda}\big(|y(0)| + |\xi(0)|\big)t \quad (44)$$

With (29), (43) and (44), an analysis similar to that for the case $\rho = 1$ can be performed to produce

$$|y(t)-y(0)| \le \delta_5(T)|y(0)| + \delta_6(T)||z(0)|| + \delta_7(T)||\xi(0)|| \quad (45)$$

where

$$\begin{aligned}\delta_5(T) &= H^{-1}(L_1 + \lambda^{-1}L_{u2} + \sigma^{-1}k_2L_{u2})\big(e^{HT}-1\big) \\ \delta_6(T) &= \sigma^{-1}k_2\big(\sigma e^{HT} + He^{-\sigma T} - (H+\sigma)\big)(H+\sigma)^{-1} \\ &\quad + \sigma^{-1}k_2\big(1-e^{-\sigma T}\big) \\ \delta_7(T) &= \lambda^{-1}(1-e^{-\lambda T}) + \lambda^{-1}L_{u2}(e^{HT}-1) \\ &\quad + \lambda^{-1}\big(\lambda e^{HT} + He^{-\lambda T} - (H+\lambda)\big)(H+\lambda)^{-1}\end{aligned}$$

The time derivative of V_{d2} during the interval $[0,T)$ satisfies

$$\begin{aligned}\dot{V}_{d2} &= -(\gamma_1+2)\|z\|^2 + 2z^{\mathrm{T}}P\phi_z + y(z_1+\phi_y+\xi) \\ &\quad + \tilde{\xi}\Big(-\lambda\xi + u_{c2} - \frac{\partial\hat{\xi}}{\partial y}(z_1+\phi_y+\xi)\Big) \\ &\quad + \tilde{\xi}(u_{d2}-u_{c2}) \\ &\le -k_1y^2 - \gamma_1||z||^2 - \lambda\tilde{\xi}^2 \\ &\quad + |\tilde{\xi}||u_{d2}-u_{c2}| \end{aligned}\quad (46)$$

In addition, it is known that $\xi = \tilde{\xi} + \hat{\xi}$ and $\hat{\xi}$ (i.e., u_{c1}) is a function of y with a Lipschitz constant L_{u1}. Then we have

$$|\xi(0)| \le |\tilde{\xi}(0)| + |\hat{\xi}(y(0))| \le |\tilde{\xi}(0)| + L_{u1}|y(0)| \quad (47)$$

From (41), (45) and (47), it can be verified that

$$|\tilde{\xi}||u_{d2}-u_{c2}| \le \varepsilon_1|y(0)|^2 + \varepsilon_2||z(0)||^2 + \varepsilon_3|\tilde{\xi}(0)|^2 + \varepsilon_4|\tilde{\xi}|^2$$

where $\varepsilon_1 = L_{u2}\big(\delta_3+\delta_5+L_{u1}(\delta_4+\delta_7)\big)$, $\varepsilon_2 = L_{u2}\delta_6$, $\varepsilon_3 = L_{u2}(\delta_4+\delta_7)$, $\varepsilon_4 = L_{u2}\big(\sum_{i=3}^7\delta_i + L_{u1}(\delta_4+\delta_7)\big)$. From above we then have

$$\begin{aligned}\dot{V}_{d2} &\le -k_1y^2 - \gamma_1||z||^2 - (\lambda-\varepsilon_4)\tilde{\xi}^2 \\ &\quad + \varepsilon_1|y(0)|^2 + +\varepsilon_2||z(0)||^2 + \varepsilon_3|\tilde{\xi}(0)|^2 \\ &\triangleq -\alpha_2(T)V_{d2} + \beta_2(T)V_{d2}(\chi(0))\end{aligned}\quad (48)$$

where

$$\begin{aligned}\alpha_2(T) &= \min\left\{2k_1, \frac{\gamma_1}{\lambda_{max}(P)}, 2(\lambda-\varepsilon_4)\right\} \\ \beta_2(T) &= \max\left\{2\varepsilon_1, \frac{\varepsilon_2}{\lambda_{min}(P)}, 2\varepsilon_3\right\}\end{aligned}\quad (49)$$

Note that $\varepsilon_i(0) = 0$ $(i = 1,2,3,4)$. Following the same approach as shown in the analysis for the case $\rho = 1$, we can establish that there exists a T_4^* so that for all $0 < T < T_4^*$, $\alpha_2(T) > \beta_2(T) > 0$. Finally, set $T_2 := \min(T_3^*, T_4^*)$, which completes the proof by Lemma 2.

Remark 1. Both Theorem 3 and Theorem 4 only declare the existence of a certain upper limit of sampling period, but states no information of the affects of control parameters and initial sets of the system on the upper limit. The way the initial set and the controller parameters affect the upper limit is still subject to further investigation.

Remark 2. If $\rho = 1$, the controller reduces to a static feedback controller. If $\rho = 2$, the dynamic controller uses a particular linear filter, a convenience in control implementation. However, stability analysis for $\rho \geq 3$ is more complicated and remains a subject of further studies.

5. SIMULATION

Consider the following system with relative degree $\rho = 2$

$$\dot{x}_1 = x_2 + y^2$$
$$\dot{x}_2 = u, \quad y = x_1$$

The filter $\dot{\xi} = -\lambda\xi + u$ is introduced so that the filtered transformation $\eta_1 = x_1$ and $\eta_2 = x_2 - \xi$, and the state transformation $z = \eta_2 - \lambda\eta_1$ can render the system into the following form

$$\dot{z} = -\lambda z + y^2 - \lambda^2 y - \lambda y^2$$
$$\dot{y} = z + (\lambda + y)y + \xi$$

Finally, the stabilizing function $\hat{\xi} = -ky - (\lambda + y)y - \frac{1}{2}\lambda^2(y+\lambda)^2 y^2$ and the control u_c can be obtained using (11). For simulation, we choose $\lambda = 3$, $k = 4$.

Simulations are carried out by Simulink using zero-order hold blocks for the case where the initial values are $x_1(0) = 1$ and $x_2(0) = 200$. Results shown in Fig.1 and Fig.2 indicate that the sampled-data system is asymptotically stable when $T = 0.0001s$. Further simulations show that the overall system is unstable if $T = 0.0005s$. In summary, the example illustrates that for a range of sampling period T, the sampled-data controller designed in the former sections can asymptotically stabilise the sampled-data system.

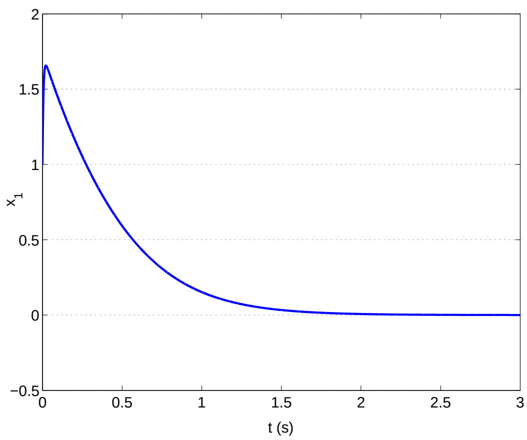

Fig. 1. The time response of x_1 for $T = 0.0001$

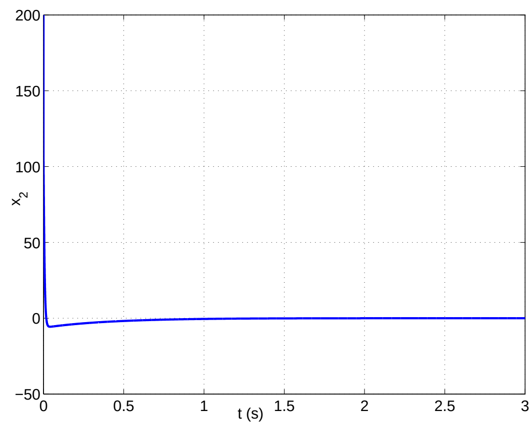

Fig. 2. The time response of x_2 for $T = 0.0001$

6. CONCLUSION

We have presented an analysis for output feedback control of one class of nonlinear sampled-data systems. It has been shown that in the case of relative degree $\rho = 1, 2$, the sampled-data version of continuous-time controllers will still asymptotically stabilise the system for any given initial sets, provided that the sampling period T is small.

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