Design of Retarded Fractional Delay Differential Systems by the Method of Inequalities

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Abstract: Methods based on numerical optimization are useful and effective in the design of control systems. This paper describes the design of retarded fractional delay differential systems (RFDDSs) by the method of inequalities, in which the design problem is formulated so that it is suitable for solution by numerical methods. This is an extension of the formulation proposed by Zakian and Al-Naib (*Proc. IEE* **120**, pp. 1421–1427, 1973) in connection with rational systems. In using the formulation with RFDDSs, the associated stability problems are resolved by using the stability test and the numerical algorithm recently developed by the authors, whereas the time-responses are obtained by using a known method for numerical inversion of Laplace transforms. Two numerical examples are given, where fractional controllers are designed, respectively, for a time-delay plant and a heat-conduction process.

Keywords: fractional systems; systems with time-delays; control systems design; the method of inequalities; parameter optimization.

1. INTRODUCTION

Recently, much research effort has been given to fractional differential systems. Many physical processes have their mathematical models described by fractional delay differential equations (see, for example, Podlubny [1999a] and the references therein). Moreover, it is demonstrated [Podlubny, 1999b] that if appropriately designed, feedback systems with fractional order controllers can yield better performances than those with integer order controllers.

Many investigators have been prompted to develop methods for designing fractional differential systems in order to enhance the system's performances and robustness (see, for example, Podlubny [1999b], Maione and Lino [2007], Zhao et al. [2005] and also the references therein). These methods are suitable for handling simple design problems with some specific design specifications. It is evident that the design problems become much more complicated when there are a number of design objectives to fulfill simultaneously. Therefore, it is desirable to have a systematic method that can solve the design problems for fractional differential systems efficaciously.

Methods based on numerical optimization have proved useful and effective in the design of control systems. The method of inequalities [Zakian and Al-Naib, 1973, Zakian, 1979b, 1996, 2005], or MoI, is a general multiobjective optimization method that requires the formulation of design problems as a set of inequalities. The method facilitates a realistic formulation of the design problem by allowing the designer to express the constraints and the performance specifications directly in terms of inequalities, whereas all tedious computations are carried out by efficient numerical algorithms. The method has been successfully applied to many difficult design problems (see, for example, Taiwo [1978, 1979, 1980], Whidborne et al. [1994], Janabi and Gray [1991], Balachandran and Chidambaram [1997] and also many references cited in Zakian [2005]). So far, none has considered the design of fractional differential systems using the MoI.

The objective of this paper is to describe the design of retarded fractional delay differential systems (RFDDSs) by the MoI, where the transfer functions of the systems are of the form

$$\frac{q_0(s) + \sum_{k=1}^{n_2} q_k(s)e^{-\beta_k s} + \sum_{k=1}^{\tilde{n}_2} \tilde{q}_k(s)e^{-v_k(s)}}{p_0(s) + \sum_{k=1}^{n_1} p_k(s)e^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}_1} \tilde{p}_k(s)e^{-u_k(s)}}, \tag{1}$$

the delays γ_k and β_k are such that $0 < \gamma_1 < \ldots < \gamma_{n_1}$ and $0 < \beta_1 < \ldots < \beta_{n_2}$, the polynomials p_k, q_k, $\tilde{p}_k$ and $\tilde{q}_k$ are of the form $\sum_{j=0}^{l_k} a_j s^{\alpha_j}$ with all $\alpha_j \geq 0$, $\deg p_0 > \deg p_k$ for $k = 1, 2, \ldots, n_1$, $\deg p_0 \geq \deg q_0$ and $\deg p_0 > \deg q_k$ for $k = 1, 2, \ldots, n_2$, u_k and v_k are polynomials of the form $\sum_{j=1}^{m_k} b_j s^{\delta_j}$ with $0 < \delta_j \leq 1$, and $b_j \geq 0$, and none of u_k and v_k assumes the form αs.

Obviously, the class of systems (1) is very general and includes rational systems and retarded delay differential systems as special cases.

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The design problem presented in the paper is formulated in such a way that it is suitable for solution by numerical methods. In fact, we extend the formulation which was first proposed by Zakian and Al-Naib [1973] in connection with rational systems, to the more general case of systems described by the transfer function in (1). In doing this, the associated stability problems are resolved by using the stability test and the numerical method for computing the abscissa of stability that have been recently developed by the authors (see Section 3.1 for the details), whereas the time-responses are obtained by using a known method for numerical inversion of Laplace transforms (see Section 3.2 for the details).

The organization of the paper is as follows. Section 2 describes the MoI and the general principle of the design formulation that was considered by Zakian and Al-Naib [1973]. In Section 3, such a formulation is extended to the case of RFDDSs in such a way that it is suitable for solution by numerical methods. In addition, the numerical methods involved are briefly described. In Section 4, two illustrative numerical examples are given, where fractional controllers are designed, respectively, for a time-delay plant and a heat conduction process. Conclusions and discussion are provided in Section 5.

2. DESIGN BY THE MOI

The MoI requires that a design problem be formulated as a set of inequalities

$$\phi_i(p) \leq C_i, \quad i = 1, 2, ..., m, \tag{2}$$

where $p \in \mathbb{R}^n$ is the design parameter vector, $\phi_i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ represent performance measures or physical properties of the system, and the bounds C_i are the supremal values of $\phi_i(p)$ that can be tolerated. Any point p satisfying (2) is an acceptable design solution.

In practice, numerical methods are employed to compute a design solution of (2). For this reason, it is important that the design problem (2) be formulated in such a way that it is suitable for solution by numerical methods.

It is noted [Zakian and Al-Naib, 1973, Zakian, 1979b, 1987, 2005] that the process of solving (2) by numerical methods involves two phases of computation.

- **Phase I:** Determine a point p such that $\phi_i(p) < \infty \ \forall i$.
- **Phase II:** Determine a solution of inequalities (2) by starting at the point obtained in Phase I.

Each phase gives rise to distinct computational problems.

2.1 Phase I

Define *a stability point* as a point p satisfying

$$\phi_i(p) < \infty \quad \forall i \tag{3}$$

and define *the stability region* Ω as the set containing all stability points. The main problem of Phase I is in the computation of a stability point.

In general, inequalities (3) are not soluble by numerical methods. That is, a stability point cannot be generated using only the functions ϕ_i and a descent method. This is because a gradient of similar property of ϕ_i cannot be defined outside the stability region Ω.

Zakian (see Zakian and Al-Naib [1973], Zakian [1987, 2005]) advocates that a possible method for obtaining a stability point by numerical methods is to replace condition (3) by an equivalent inequality

$$\alpha(p) < 0 \tag{4}$$

such that (i) $\alpha(p) < \infty$ for all $p \in \mathbb{R}^n$ and (ii) α can be computed economically in practice.

Once there exists such a function α, condition (4) becomes soluble by numerical methods. Consequently, iterative numerical methods can be used to locate a stability point by starting from any arbitrary point in $\mathbb{R}^n$.

For rational systems, Zakian and Al-Naib [1973] chose α to be the abscissa of stability of a characteristic polynomial. In this connection, an economical algorithm for computing the abscissa of stability was given in Zakian [1979a]. This useful approach was extended later to the case of retarded delay differential systems by Arunsawatwong [1994, 1996] and to the case of RFDDSs by Nguyen and Arunsawatwong [2008]. See also Section 3 for further details.

Hence, for RFDDSs (including rational and retarded delay differential systems), (3) is replaced by

$$\alpha(p) < 0, \quad \alpha \triangleq \sup\{\operatorname{Re} s : \ f(s) = 0\}, \tag{5}$$

where $f(s)$ denotes the characteristic function of the system. Usually, the inequality (5) is replaced by a practical sufficient condition

$$\alpha(p) \leq -\varepsilon \quad (\varepsilon > 0). \tag{6}$$

It should be noted that the bound $-\varepsilon$ is introduced in (6) so as to ensure that the system is stable as long as the magnitude of error in the computed value of $\alpha(p)$ is less than ε.

2.2 Phase II

By starting from a stability point, a search method locates a solution of (2) within the stability region Ω. To avoid any risk in stepping outside the stability region, one checks the stability of the system at every point generated by the search method. In doing this, one needs to determine the sign of $\alpha(p)$, which is less demanding than to compute the value of $\alpha(p)$. Therefore, in Phase II, only a stability test is required.

Once the system is found to be stable, the performance $\phi_i(p)$ are to be computed. In contrast to Phase I, the main problem in Phase II is in computing the functions ϕ_i economically for given values of p. Usually, the functions ϕ_i that are the most difficult to compute are those defined in terms of the time-responses of the system. Once the time-responses are obtained, such functions ϕ_i can be computed using known numerical methods (see Section 4 for this). Therefore, it is evident that one needs an efficient and reliable algorithm for computing the time-responses.

3. DESIGN FORMULATION FOR RFDDS

Following the general principle given in Section 2, this section describes a practical formulation for the design of RFDDSs by the MoI, which is suitable for solution by numerical methods. In addition, the computational algorithms involved are briefly explained.

3.1 Stability Test & Stabilization

It is known [Bonnet and Partington, 2001] that a system characterized by the transfer function (1) is BIBO stable if and only if the characteristic function

$$f(s) = p_0(s) + \sum_{k=1}^{n_1} p_k(s) e^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}_1} \tilde{p}_k(s) e^{-u_k(s)} \qquad (7)$$

has all zeros with negative real parts. Accordingly, an RFDDS is stable if and only if

$$\alpha < 0, \quad \alpha \triangleq \sup\{\mathrm{Re}\, s : \ f(s) = 0\}. \qquad (8)$$

Recently, Nguyen and Arunsawatwong [2008] have developed a computational stability test and a practical algorithm for computing the abscissa of stability for RFDDSs. The algorithm makes repeated use of the stability test and thereby avoids computing of all zeros of the characteristic function $f(s)$.

For RFDDSs, once the abscissa of stability α can be efficiently computed, a stability point is readily obtainable by simply solving the inequality (6) by iterative numerical methods.

Stability Test Let $H(\rho)$ be a right half plane and described by

$$H(\rho) \triangleq \{s \in \mathbb{C} : \mathrm{Re}(s) \geq \rho\},$$

where $\rho \in \mathbb{R}$ is specified. An RFDDS is said to be $H(\rho)$-*stable* if none of its characteristic roots lies in $H(\rho)$.

By modifying the stability test due to Hwang and Cheng [2006], Nguyen and Arunsawatwong [2008] devise a numerical procedure for testing the $H(\rho)$-stability for any given number ρ, which is more general. The key idea of the $H(\rho)$-stability test is briefly summarized as follows.

The half plane $H(\rho)$ is represented by a semicircle with infinite radius and the well-known Cauchy's residue theorem is applied to determine whether the characteristic function $f(s)$ has no zeros in $H(\rho)$.

Notice that in general, $f(s)$ in (7) has a branch cut along the negative real axis. Therefore, when $\rho \leq 0$, the contour is indented to avoid crossing the branch cut along the negative real axis. In which case, the search for any roots of the characteristic function $f(s)$ on the interval $[\rho, 0]$ can be performed readily by using one dimensional search methods.

In Phase II, neither should the search method step outside Ω nor it should generate trial points very close to the boundary of the stability region Ω. This is because of the requirement (6) and because, if the design problem is properly formulated, a design solution usually lies well inside the stability region Ω. For these reasons, in Phase II, it is advisable to perform an $H(-\varepsilon)$-stability test with $\varepsilon > 0$ sufficiently small, instead of performing an $H(0)$-stability test.

Notice that the system is $H(-\varepsilon)$-stable if and only if condition (6) is satisfied. Hence, the $H(-\varepsilon)$-stability test is used for determining whether $\alpha(p) \leq -\varepsilon$ for a given p.

Computation of Abscissa of Stability The numerical method for computing the abscissa of stability for RFDDSs is devised by modifying Zakian's (1979a) algorithm, which is a bisection method and makes repeated use of the $H(\rho)$-stability test.

In essence, $\alpha(p)$ is computed by the following bisection algorithm. First, determine two numbers a and b such that $\alpha(p) \in (a, b)$; that is, the system is both $H(a)$-unstable and $H(b)$-stable. Compute the midpoint $c = (a+b)/2$ and perform an $H(c)$-stability test so as to determine which of the two intervals (a, c) and (c, b) contains $\alpha(p)$. Repeat the bisection until the interval containing $\alpha(p)$ is sufficiently small.

The details of the algorithm can be found in Nguyen and Arunsawatwong [2008].

3.2 Computation of Time Responses

In this paper, we compute the time-responses of RFDDSs by employing Zakian's I_{MN} approximants [Zakian, 1969, 1975], which result in a useful formula for numerically inverting Laplace transforms (see also Taiwo et al. [1995], Arunsawatwong [1998]).

Definition of I_{MN} Approximants Zakian [1969, 1975] defines the I_{MN} approximant of $x(t)$ for $t \geq 0$ by the improper integral

$$I_{MN}(x, t) \triangleq \int_0^\infty x(\lambda t) \sum_{i=1}^{N} K_i \mathrm{e}^{-\alpha_i \lambda} d\lambda, \qquad (9)$$

where (α_i, K_i) are defined constants, and the nonnegative integers (M, N) are, respectively, the orders of the numerator and the denominator of the Laplace transform of $\sum_{i=1}^{N} K_i \mathrm{e}^{-\alpha_i \lambda}$.

In this work, we restrict our attention only to the *full grade* I_{MN} approximants [Zakian, 1975], whose constants (α_i, K_i) are defined by

$$\sum_{i=1}^{N} \frac{K_i}{z + \alpha_i} = \mathrm{e}_{MN}^{-z} \quad \text{and} \quad \mathrm{Re}(\alpha_i) > 0 \; \forall i, \qquad (10)$$

where e_{MN}^{-z} denotes the $[M/N]$ Padé approximant to e^{-z}. The *full grade* I_{MN} approximants have many remarkably useful properties [Zakian, 1975, Arunsawatwong, 1998] and have been successfully applied to many practical problems (see, for example, Zakian and Al-Naib [1973], Taiwo et al. [1995], Rice and Do [1994]).

Inversion Formula Let $X(s)$ denote the Laplace transform of $x(t)$, evaluated at s. That is,

$$X(s) \triangleq \mathcal{L}[x(t)] = \int_0^\infty x(t) \mathrm{e}^{-st} dt, \qquad (11)$$

where s is a complex number such that the integral converges to a finite limit. From (9), it can be readily verified Zakian [1969] that

$$I_{MN}(x, t) = \frac{1}{t} \sum_{i=1}^{N} K_i X\Big(\frac{\alpha_i}{t}\Big), \quad t > 0. \qquad (12)$$

Evidently, (12) provides a useful formula for the numerical inversion of Laplace transforms. In this paper, we normally use $M = 11$ and $N = 18$ with double precision arithmetic operations. However, whenever there is a doubt in the accuracy of the obtained results, we recompute by using $M = 30$ and $N = 40$ with quad-precision arithmetic operations. The details of how to choose appropriate values of M and N for the inversion formula (12) can be found in Taiwo et al. [1995].

4. NUMERICAL EXAMPLES

This section demonstrates how to design RFDDSs by the MoI using the formulation described in Section 2 and the algorithms mentioned in Section 3. For clarity, we focus our attention only on the design of SISO systems. However, it is important to note that the systematic method developed in the paper can readily be applied to MIMO systems of fractional order.

4.1 Design Formulation & Specifications

Consider an SISO control system shown in Fig. 1, where $G(s)$ is the plant transfer function, $K(s, p)$ is the controller transfer function with design parameter p. Suppose that the reference r is a unit step function.

The design parameter p is to be determined so that it satisfies the following design specifications.

$$\phi_i(p) \le C_i, \quad i = 1, 2, 3, 4, \tag{13}$$

where

$$\left.\begin{aligned} \phi_1 &\triangleq \sup_{t\ge 0} \big[y(t) - y_\infty\big]/y_\infty \\ \phi_2 &\triangleq \min\{t : y(t) = 0.9\, y_\infty\} \\ \phi_3 &\triangleq \min\{\tau : |y(t) - y_\infty| \le 0.02\, y_\infty \;\forall\, t \ge \tau\} \\ \phi_4 &\triangleq \sup_{t\ge 0} |u(t)| \end{aligned}\right\}, \tag{14}$$

and y_∞ denotes the steady state value of the step response y, that is,

$$y_\infty \triangleq \lim_{t\to\infty} y(t).$$

In many practical applications, the actuator has a saturation characteristic. To avoid the saturation during the operation of the system, the design requirement $\phi_4 \le C_4$ in (13) needs to be taken into design consideration. This fundamental requirement makes the design problem become difficult to solve even for an SISO system. However, it should be noted that the MoI can solve such a design problem effectively in a systematic manner (see, for example, Taiwo [1978, 1979]).

In addition to the design specifications in (13), if there are constraints on the design parameter p (or permissible ranges of p), they can be incorporated into the design inequalities (13) very easily. This results in the inequalities of the form (2).

Throughout this work, the design inequalities (13) or (2) are solved by using a numerical search algorithm called *the moving boundaries process* (MBP). See Zakian and Al-Naib [1973] for the details of the algorithm. For the efficiency in the design process, tasks such as the $H(\rho)$-stability test, the computation of $\alpha(p)$ and the step responses are implemented in FORTRAN programming language.

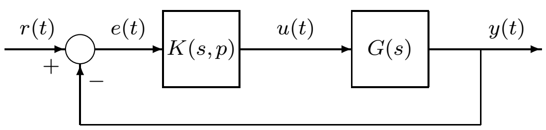

Fig. 1. A Unity Feedback Control System.

4.2 Controller Design for a Time-Delay Plant

Let the plant transfer function $G(s)$ be given by

$$G(s) = \frac{2e^{-2s}}{(s+1)(s+2)}. \tag{15}$$

Assume that a fractional PI controller is used and the controller transfer function $K(s, p)$ is

$$K(s, p) = \frac{p_1 + p_2 s^{p_3}}{s^{p_3}}, \tag{16}$$

where $p = [p_1,\ p_2,\ p_3]^T$ is the vector of design parameters that are constrained to be positive.

It is easy to verify that the transfer function of the closed-loop system takes the form (1) and is given by

$$H(s) = \frac{2(p_1 + p_2 s^{p_3})e^{-2s}}{s^{p_3}(s+1)(s+2) + 2(p_1 + p_2 s^{p_3})e^{-2s}}. \tag{17}$$

Suppose that the bounds C_i are specified as follows.

$$C_1 = 0.05,\ C_2 = 5.7,\ C_3 = 6.5,\ C_4 = 1.1. \tag{18}$$

By starting at the point $p_0 = [0.23,\ 0.49,\ 1.0]^T$, for which $\alpha(p_0) = -0.2665$, the MBP algorithm locates a design solution

$$p = [0.225,\ 0.491,\ 1.043]^T, \tag{19}$$

where $\alpha(p) = -0.2714$ and the corresponding performance measures are

$$\phi_1 = 0.02,\ \phi_2 = 5.62,\ \phi_3 = 6.37,\ \phi_4 = 1.06.$$

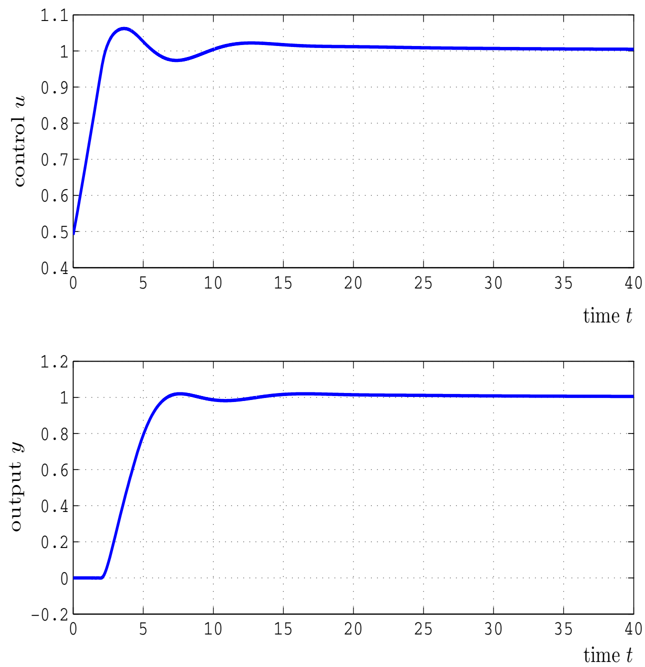

Fig. 2. Step responses of delay system

The control and output signals of the system with the controller parameter p in (19) are shown in Fig. 2.

4.3 Controller Design for a Heat-Conduction Process

Consider the plant whose transfer function $G(s)$ is given by

$$G(s) = \frac{1}{\sqrt{s}\,\sinh(\sqrt{s}\,)}. \tag{20}$$

The nonrational transfer function in (20) occurs when the plant is governed by a heat conduction (or diffusion) equation (see, for example, Schwarz and Friedland [1965], Rice and Do [1994]).

Assume that a fractional phase-lead controller is used and its transfer function is

$$K(s,p) = \frac{p_1(s^{p_4}+p_2)}{(s^{p_4}+p_3)}, \tag{21}$$

where $p_3 > p_2 > 0$ and $p_4 > 0$. It is worth noting [Podlubny et al., 2002] that the controller in (21) can be realized in practice.

The closed-loop transfer function takes the form (1) and is given by

$$H(s) = \frac{2p_1(s^{p_4}+p_2)e^{-\sqrt{s}}}{\begin{pmatrix}\sqrt{s}(s^{p_4}+p_3)(1-e^{-2\sqrt{s}}) \\ +\,2p_1(s^{p_4}+p_2)e^{-\sqrt{s}}\end{pmatrix}}. \tag{22}$$

Suppose that the bounds C_i are specified as follows.

$$C_1 = 0.05,\ C_2 = 0.35,\ C_3 = 0.4,\ C_4 = 10.0. \tag{23}$$

Starting at the point $p_0 = [9.2,\ 7.5,\ 15,\ 1.1]^T$, for which $\alpha(p_0) = -4.4694$, the following design solution is found by the MBP algorithm.

$$p = [9.240,\ 7.513,\ 15.204,\ 1.101]^T, \tag{24}$$

where $\alpha(p) = -4.4383$ and the corresponding performance measures are

$$\phi_1 = 0.02,\ \phi_2 = 0.34,\ \phi_3 = 0.39,\ \phi_4 = 9.24.$$

The control and output signals of the system with the controller parameter p in (24) are shown in Fig. 3.

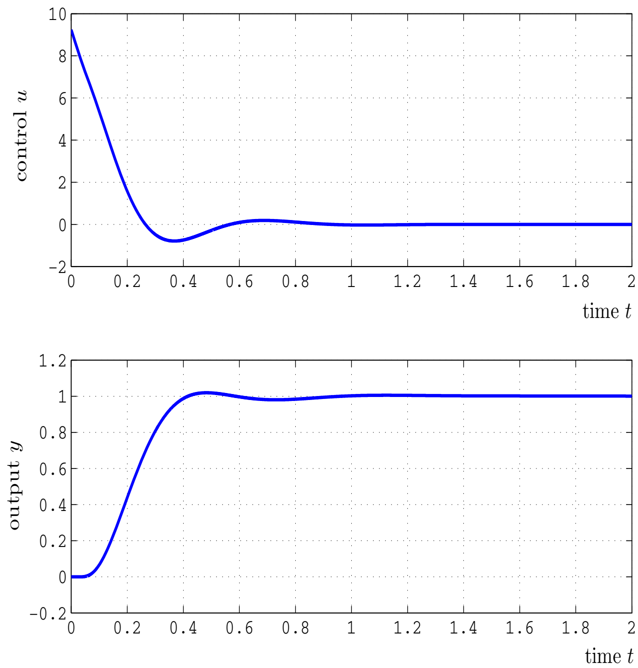

Fig. 3. Step responses of heat conduction system

5. CONCLUSIONS AND DISCUSSION

This paper describes the design of RFDDSs by the MoI, in which the design problem is formulated so that it is suitable for solution by numerical methods. This is an extension of the design formulation which was first used by Zakian and Al-Naib [1973] in conjunction with rational systems and subsequently by Arunsawatwong [1994, 1996] in conjunction with retarded delay differential systems.

The use of this formulation is made possible for RFDDSs because the associated stability problems have been resolved by using the stability test and the algorithm for computing the abscissa of stability which have been recently developed by Nguyen and Arunsawatwong [2008]. Moreover, in this formulation, the time responses of the system are obtained efficiently by using the Laplace transform inversion formula based on Zakian's I_{MN} approximants. Once the time-responses are obtained, the performances defined in terms of the responses (which are usually the most difficult ones to compute) are easily obtainable by known numerical algorithms.

The numerical results evidently show that by using the MoI, one can design RFDDSs effectively in a systematic way. Consequently, one can deal more easily with a sophisticated design problem and, provided that appropriate design criteria are used, can arrive at an accurate and realistic formulation of the design problem.

It is of interest to note that the design formulation for RFDDSs presented in this paper is not only useful for the MoI but also for other numerical optimization methods that search for a solution in a design-parameter space.

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