Phase Model for the relaxed van der Pol oscillator and its application to synchronization analysis ⋆

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Abstract: A one dimensional phase model for the classic two dimensional van der Pol oscillator is developed. This model is restricted to the relaxed case, and its construction is based on the slow and fast transitions the phase goes through during its cycle. An application of the phase model is included in which synchronization of two coupled van der Pol oscillators is analyzed and even used to calculate the coupling strength needed for their synchronization.

1. INTRODUCTION

Synchronization has caught the attention of scientists for a long time, evidence of this is the first observation and description of this phenomenon contributed by Dutch researcher Christiaan Huygens in 1665. After him many other scientists such as Appleton, van der Pol and Peskin made experiments in living and non living systems as well as theoretical studies regarding the synchronization phenomenon (Blekhman [1988],Pikovsky et al. [2001],Strogatz and Stewart [1993]).

In (Winfree [1990]) the phase models of oscillators where introduced to study the synchronization phenomena. Phase models consist in describing the dynamics of the phase of the oscillator with an ordinary differential equation and assuming that the magnitude of oscillation keeps constant. Disadvantages of usual phase models resides in the fact that there is no relationship between the original two dimensional oscillator, van der Pol or pendulum-like, and the associated phase model. This paper fulfills this gap constructing a piecewise linear phase model associated to a relaxed type van der Pol oscillator.

Regarding synchronization, the advantages of 1-dimensional models is that if we can show that the phase difference is bounded by a small number, the synchronization is proved. There resides the relevance of the developed model to predict synchronization of two coupled van der Pol oscillators.

The organization of this paper is: Section II gives some preliminaries on phase models and synchronization, in Section III we describe the van der Pol oscillator and a convenient change of variable is done, in Section IV the phase model for the van der Pol oscillator is constructed, in this section we also compare the constructed phase model with the complete two dimensional model. In Section V we use the phase model proposed to observe the synchronization of two coupled van der Pol oscillators, analysis is done on the phase model to obtain the strength coupling for attaining synchronization in the complete model. Finally in Section V the conclusions are stated.

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2. PRELIMINARIES

2.1 Phase Models

The state of a system consists of everything you need to know about it in order to know what it will do next in response to a stimulus. The state is determined by a collection of state variables, which can be many for each system, but for specific interest some of them are more important than the rest (Winfree [1990]).

In the case of oscillators, one can exploit a separation of timescales: on a fast one the oscillator relaxes to its limit cycle and on a slow one the phase evolves (Strogatz [2000]), therefore when choosing the important state variable as Winfree suggests, the oscillator can be characterized solely by its phase.

Considering a general two-dimensional autonomous system of ordinary differential equations:

$$\frac{d\mathsf{x}}{dt} = \mathsf{f}_0(\mathsf{x}),\ \mathsf{x} = (x_1, x_2) \qquad (1)$$

with a stable periodic solution $\mathsf{x}_0(t) = x_0(t + T_0)$, the plot of this solution in the phase plane as an isolated closed attractive trajectory known as limit cycle. A point in the phase plane moving along the cycle represents the self-sustained oscillations.

Introducing the phase ϕ as a coordinate along the limit cycle, the system can be described by:

$$\phi = f_1(t) \qquad (2)$$

where the natural frequency of the self-sustained oscillations is in average $\omega_0 = \frac{2\pi}{T_0}$, but can also show some time dependency (Pikovsky et al. [2001]).

Then a phase model of system (1) is the equation (2), where the standing out state variable is the phase.

2.2 Synchronization

There is not yet a unified definition of synchronization, we consider the following one which combines those in (Izhikevich [2006],Pikovsky et al. [2001],Rosenblum and Pikovsky [2003]).

Definition 1. Synchronization is the adjustment in the frequency of two or more coupled oscillators due to the existence of an interaction between them; this adjustment pronounces in a collective behavior that is not intrinsic to any individual oscillator. Synchronization is essentially a nonlinear effect that happens solely in the self-sustained systems (Mimila [2006]).

The coupling existing between oscillators can be either unidirectional or bidirectional, in the latter case it can be symmetric if the coupling strength is the same for both oscillators, or asymmetric if the coupling strength is different for each oscillator. The unidirectional coupling corresponds to a forced oscillator which does not influence the forcing oscillator and therefore a system of this kind does not fulfill the conditions for synchronization (Izhikevich [2006]).

Once a pair of oscillators are coupled and attain a common behavior they are said to be frequency locked if they share the oscillating frequency, and they are phase locked if there remains a constant phase difference between them (Izhikevich [2006]).

Synchronization can be of different types: in phase if they do not have any phase difference; in anti-phase if they have a phase difference of half a period, and out of phase if they have a phase difference other than half a period (Izhikevich [2006]).

Determining the synchronization of two coupled oscillators can not always be easy in the time domain, in this case the stroboscopic observation can be helpful. This method consists in observing the phase plane of the oscillators not continuously but for times $t_k = kT$, $k = 1, 2, 3, \ldots$ and T the oscillation period of one of the observed oscillators. We say there is synchronization when the oscillators are successively plotted in the same point, indicating that they both have reached the sampling period, therefore they share the frequency (Pikovsky et al. [2001],Strogatz [2003]).

Stroboscopic observation is equivalent to the Poincaré map for the case in which the oscillators are isochronous, i.e., their frequency is independent of the amplitude.

3. VAN DER POL OSCILLATOR

The well known equation of the Van der Pol oscillator is (van der Pol [1926]):

$$\ddot{x} + \varepsilon\omega\left(x^2 - 1\right)\dot{x} + \omega^2 x = 0 \tag{3}$$

The main interest for this paper is the relaxation case, i.e. $\varepsilon \gg 1$, and for such case the trajectory of the solution of this equation in the phase plane $(x, \dot{x})$ becomes quite large even for a small frequency value ω. The decision to change to Lienard's coordinates is made given that in this coordinates the trajectory in the phase plane doesn't increase largely as ε increases.

The change to Lienard's coordinates is $y \triangleq \dot{x} + \varepsilon\omega\left(\frac{x^3}{3} - x\right)$ (Khalil [2002]). Once the change of variable is done, the following van der Pol model is obtained:

$$\left.\begin{aligned} \frac{dx}{dt} &= y - \varepsilon\omega\left(\frac{x^3}{3} - x\right) \\ \frac{dy}{dt} &= -\omega^2 x \\ x(0) &= x_0; \dot{x}(0) = \dot{x}_0 \\ y(0) &= \dot{x}_0 + \varepsilon\omega\left(\frac{x_0^3}{3} - x_0\right) \end{aligned}\right\} \tag{4}$$

Remark 1. For $\varepsilon \gg 1$ the phase of the model (3) is essentially 0 or π with very fast transition between these two values. For the model (4) the change has several well identified fast regions and some other slow regions. See figures 1,2 and 3.

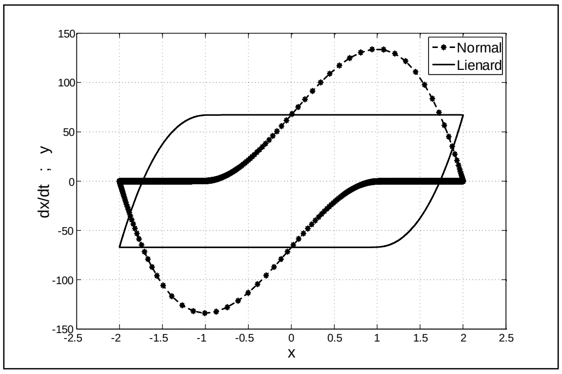

Fig. 1. Phase Plane of the relaxation van der Pol Oscillator in normal (dotted line) and Lienard's (solid line) coordinates.

In order to keep track of the dynamics of the phase, a change to polar coordinates is made considering the following definitions:

$$\left.\begin{aligned} r &\triangleq \sqrt{y^2 + \omega^2 x^2} \\ \theta &\triangleq \arctan\left(\frac{y}{\omega x}\right) \end{aligned}\right\}; \; i.e. \; \begin{aligned} x &= \frac{r}{\omega}\cos\theta \\ y &= r\sin\theta \end{aligned}$$

After making this change of variables the Lienard's coordinates model (4) is expressed into the following polar coordinates model:

$$\left.\begin{aligned} \frac{d\theta}{dt} &= -\omega - \tfrac{1}{2}\varepsilon\omega\sin 2\theta + \frac{\varepsilon r^2}{24\omega}\left(\sin 4\theta + 2\sin 2\theta\right) \\ \frac{dr}{dt} &= \varepsilon\omega r\cos^2\theta - \frac{\varepsilon r^3}{3\omega}\cos^4\theta \\ r(0) &= r_0 = \sqrt{y_0^2 + \omega^2 x_0^2} \\ \theta(0) &= \theta_0 = \arctan\left(\frac{y_0}{\omega x_0}\right) \end{aligned}\right\} \tag{5}$$

Remark 2. If $\varepsilon = 0$, we get $\dot{\theta} = -\omega$ and $\dot{r} = 0$, the well known harmonic oscillator.

4. PHASE MODEL CONSTRUCTION

Model (5) was simulated for several values of ε in the interval $(0, 150]$, the obtained data is plotted in Fig. 2 where they are shown for a normalized period $T = 1$.

4.1 Motivation

From the plots in Fig. 2 it can be seen that for small values of ε the dynamics of the phase is almost linear, and as ε increases the

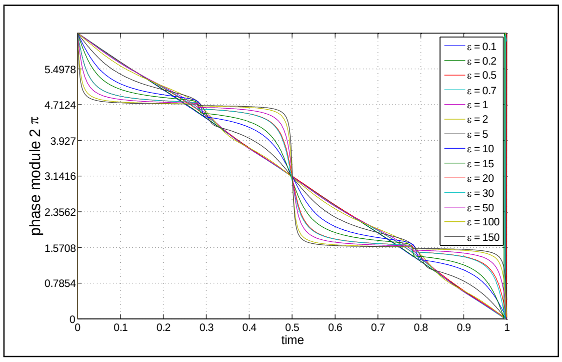

Fig. 2. Phase for several ε's; normalized period.

linearity is lost and nine different regions can be distinguished, these regions can be approximated by linear ones. Because of this, the motivation arises to construct a piecewise linear dynamics model of the phase.

In Fig. 3 we show the phase dynamics of the van der Pol oscillator for $\varepsilon = 20$ and the nine different linear regions that can be associated to it.

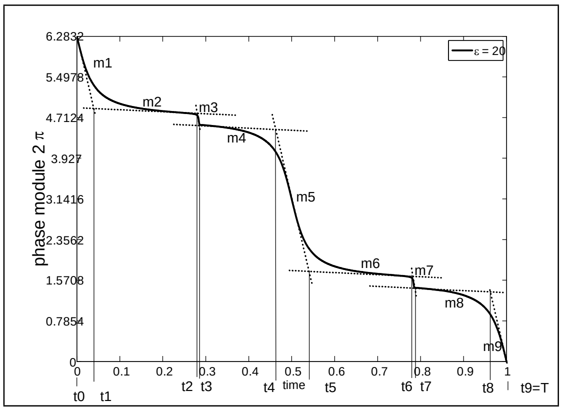

Fig. 3. Phase for $\varepsilon = 20$ and the associated linear regions.

Regardless the value of ε the following is observed:

$$m_1 = m_5 = m_9 \qquad (6)$$
$$m_3 = m_7 > m_1$$

$$m_2 = m_6 \qquad (7)$$
$$m_4 = m_8 \qquad (8)$$

Equations (6)-(8) are also evident from the phase plane graph in Fig. 1 where the trajectory of the van der Pol oscillator shows a clear symmetry.

This means that the dynamics of the phase is the succession of five slopes ($m_1 - m_5$) during half period and again ($m_1 - m_5$), given (6)-(8), during the second half of the period. This apparently defines ten regions, but the last slope (m_5) of the first part coincides with the first slope of the second part, and therefore only nine regions are defined.

This fact means that in order to construct the dynamics of the phase it is enough to determine the four main slopes: m_3, m_4, m_5 and m_6. To do so the following time instants are considered:

$$t_2 \quad ; \quad t_3 \quad ; \quad t_4 \quad ; \quad t_5 \quad ; \quad t_6 = t_2 + \frac{T}{2} \qquad (9)$$

being each t_i the point where the extrapolated subsequent main slopes cross. These t_i's are the normalized instants in which the transition from slope m_{i-1} to slope m_i takes place, as shown in the following definitions; (see Fig. 3):

$$\begin{aligned} m_3 &\triangleq \frac{\theta(t_2) - \theta(t_3)}{t_2 - t_3} \quad ; \quad m_4 \triangleq \frac{\theta(t_3) - \theta(t_4)}{t_3 - t_4} \\ m_5 &\triangleq \frac{\theta(t_4) - \theta(t_5)}{t_4 - t_5} \quad ; \quad m_6 \triangleq \frac{\theta(t_5) - \theta(t_6)}{t_5 - t_6} \end{aligned} \qquad (10)$$

Given (6), (7) and (8) the dynamics of the phase can be constructed as a piecewise linear T-periodic function as:

$$\theta_{ph} = \begin{cases} m_1 & ; \; 0 = t_0 < t \le t_1 \\ m_2 & ; \; t_1 < t \le t_2 \\ m_3 & ; \; t_2 < t \le t_3 \\ m_4 & ; \; t_3 < t \le t_4 \\ m_5 & ; \; t_4 < t \le t_5 \\ m_6 & ; \; t_5 < t \le t_6 \\ m_7 & ; \; t_6 < t \le t_7 \\ m_8 & ; \; t_7 < t \le t_8 \\ m_9 & ; \; t_8 < t \le t_9 = T \end{cases} \qquad (11)$$

for which the additional time instants are defined as:

$$\begin{aligned} t_1 &\triangleq t_5 - \tfrac{T}{2} \quad ; \quad t_7 \triangleq t_3 + \tfrac{T}{2} \\ t_8 &\triangleq t_4 + \tfrac{T}{2} \quad ; \quad t_9 \triangleq t_5 + \tfrac{T}{2} \end{aligned} \qquad (12)$$

Remark 3. For all ε $m_i < 0$; therefore the solution of (11) is strictly decreasing.

4.2 Effect of ε and ω on the phase model

As we can notice in (9) and (12) the normalized time instants in the phase model (11) required for slope transitions depend on the value of the oscillating period T, which depends of ε and ω, therefore a function $T(\varepsilon, \omega)$ must be constructed.

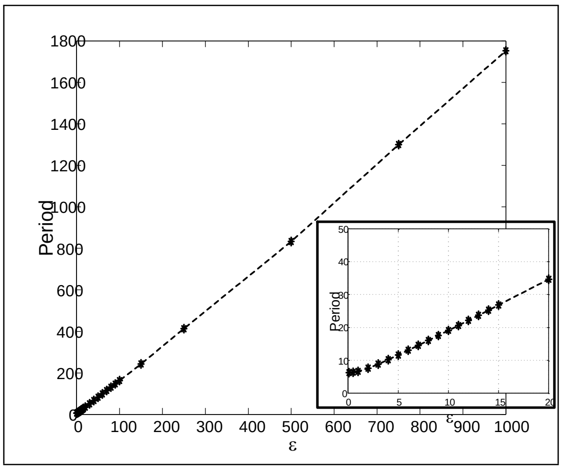

Fig. 4. Period of oscillation $T(\varepsilon)$ for $\omega = 1$.

Through simulations the period T was determined for different values of $\varepsilon \in [0, 150]$ having a fixed natural frequency $\omega = 1$, these values of $T(\varepsilon)$ are plotted in Fig. 4 where we can notice the linear behavior of the period for almost all the values of ε except those close to zero (as shown in the zoom).

Regarding the effect of ω on the period of oscillation, the well know relation $T(\varepsilon,\omega) = C\frac{2\pi}{\omega}$ prevails; $C = T(\varepsilon)$ is the function describing the behavior in Fig. 4.

We approximate that behavior through a piecewise linear function interpolating the periods obtained from simulations for $\epsilon = \{0.1, 0.5, 1, 2, 3, \ldots, 10, \ldots 20, 30, \ldots 100, \ldots\}$, where each element in ϵ is denoted as ε_k, for $k = 1, \ldots, 32$.

The approximation is a linear function with slope $m_k = \frac{T(\varepsilon_{k+1}) - T(\varepsilon_k)}{\varepsilon_{k+1} - \varepsilon_k}$ in each k^{th}-interval of $\varepsilon \in (0.1, 150)$.

The general function is given by:

$$T(\varepsilon,\omega) = \begin{cases} \frac{2\pi}{\omega}; & \varepsilon \leq 0.1 \\ m_k(\varepsilon - \varepsilon_k) + T(\varepsilon_k); & \varepsilon \in (\varepsilon_k, \varepsilon_{k+1}) \\ 1.62\frac{\varepsilon}{\omega}; & \varepsilon \geq 150 \end{cases} \quad (13)$$

where $T(\varepsilon_k)$ is known from simulations.

Remark 4. This function is a very close approximation of the value obtained through simulations, differing at most in 0.05% for $5 < \varepsilon < 35$.

The expression (13) of $T(\varepsilon,\omega)$ for the relaxation case ($\varepsilon \geq 150$) was already obtained in (van der Pol [1934]) just with a slight difference: $T(\varepsilon,\omega) = 1.61\frac{\varepsilon}{\omega}$.

There is also an approximation of the period of oscillation made in (Dorodnicyn [1953]) for the relaxation case with natural frequency $\omega = 1$, this formulae is more accurate than (13) for $\varepsilon > 17.5$ because it includes more terms.

4.3 Comparison: Phase Model vs Complete Model

The phase model proposed for a relaxed van der Pol oscillator is $\dot{\theta}_{ph} = m_i(T(\varepsilon,\omega))$ where $T(\varepsilon,\omega)$ is given by (13). We have constructed phase models for values of $\varepsilon = \{5, 10, 15, 20, 30, 50, 10, 150\}$. We illustrate here for a natural frequency of $\omega = 1$ and $\varepsilon = 5$:

$$\dot{\theta}_{ph} = \begin{cases} m_1 = -11.3755016 & ; \quad t_0 < t \leq t_1 \\ m_2 = -2.66787117 & ; \quad t_1 < t \leq t_2 \\ m_3 = -18.741479 & ; \quad t_2 < t \leq t_3 \\ m_4 = -3.53977443 & ; \quad t_3 < t \leq t_4 \\ m_5 = -11.3755016 & ; \quad t_4 < t \leq t_5 \\ m_6 = -2.66787117 & ; \quad t_5 < t \leq t_6 \\ m_7 = -18.741479 & ; \quad t_6 < t \leq t_7 \\ m_8 = -3.53977443 & ; \quad t_7 < t \leq t_8 \\ m_9 = -11.3755016 & ; \quad t_8 < t \leq t_9 = T \end{cases}$$

$$\begin{array}{lcl} t_0 = 0 & ; & t_1 = 0.87937 \\ t_2 = 3.2866 & ; & t_3 = 3.6357 \\ t_4 = 5.062 & ; & t_5 = 6.6855 \\ t_6 = 9.0927 & ; & t_7 = 9.4418 \\ t_8 = 10.868 & ; & t_9 = T = 11.6122 \end{array}$$

The phase dynamics obtained with the proposed model is plotted in Fig. 5.**a** as well as the one for the equivalent complete model.

We can see that the phase model solution is very close to the complete model solution, in order to compare them we define the error $e \, , \, \theta_c - \theta_{ph}$, and plot it in Fig. 5.**b** for one period of oscillation.

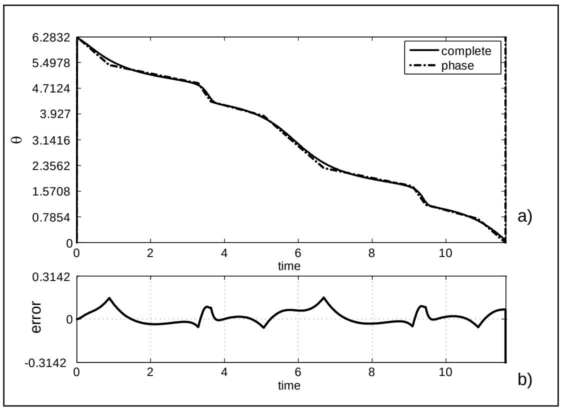

Fig. 5. a) Complete model (solid line) and Phase model(dotted line) for $\varepsilon = 5$ and $\omega = 1$. b) aproximation error $e \, , \, \theta_c - \theta_{ph}$.

Clearly the error increases when the phase has a transition from a small slope to a big one, this is a consequence of the phase model piecewise linear construction given that, when the complete model solution makes these transitions softly the phase model makes steep changes some instants later.

Also we have noticed that the error grows up as the oscillating cycles go on, this has been directly associated to an inaccurate expression for $T(\varepsilon,\omega)$ which translates in a time delay (phase shift) between the phase model solution and the complete model solution.

Remark 5. All of the observations done for the illustrated phase model with $\varepsilon = 5$ are also present for the other constructed models with $\varepsilon = \{10, 15, 20, 30, 50, 100, 150\}$, therefore in order to reduce the error of the proposed phase model a more accurate expression for the oscillating period $T(\varepsilon,\omega)$ must be used.

5. APPLICATION OF PHASE MODEL: SYNCHRONIZATION OF TWO RELAXED VAN DER POL OSCILLATORS

Considering two similar relaxed van der Pol oscillators which only differ slightly in their natural frequency, both are expressed in terms of the phase model (11) and coupled by means of $k(\theta_q - \theta_p)$ in the differential equation of p^{th} oscillator:

$$\begin{array}{c} \dot{\theta}_1 = m_{1,i}(\varepsilon,\omega_1) + k_1(\theta_2 - \theta_1); t_{1,i-1} < t < t_{1,i} \\ \dot{\theta}_2 = m_{2i,}(\varepsilon,\omega_2) + k_2(\theta_1 - \theta_2); t_{2,i-1} < t < t_{2,i} \\ nT_j(\varepsilon,\omega_j) < t_{j,i-1} < [n+1]T_j(\varepsilon,\omega_j) \\ i = 1, \ldots, 9; \; n = 0, 1, \ldots; \; j = 1, 2 \end{array} \quad (14)$$

Since $\omega_1 \approx \omega_2$ the dynamics of both phases are close to each other, and introducing enough coupling strength should make them oscillate to the same frequency with a constant phase difference.

We considering a symmetric bilateral coupling strength between both oscillators, i.e. $k_1 = k_2 \neq 0$. Stroboscopic plots of the phase of both oscillators are shown in Fig. 6, these were obtained with a sampling period $T_{av} \triangleq \frac{T_1+T_2}{2}$. All of these plots show some soft *movement*, this is due to T_{av} which isn't exactly the synchronization period.

From plot **a)** it is clear that both oscillators are close to each other when a soft coupling strength exists; from **b)** it can be seen that increasing coupling strength reduces $\Delta\theta$; finally from **c)** it can be seen that further increase in the coupling strength reduces even more the phase difference $\Delta\theta$.

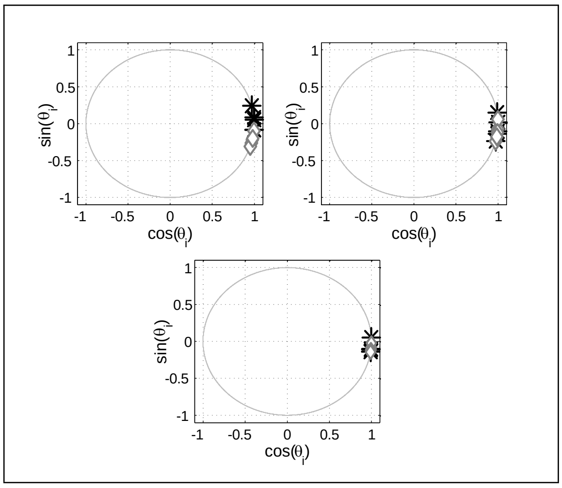

Fig. 6. Phase Model symmetrical bilateral coupling: Phase Plane for $\Delta\omega = \omega_1 - \omega_2 = 0.1$; a) $k = 0.1$, b) $k = 0.5$, c) $k = 10$

In order to compare the synchronization attained with the phase model and synchronization reached with the complete model, the system of two symmetrically coupled oscillators expressed with the complete model (5) is coupled similarly as in (14). This system is stroboscopically plotted in Fig. 7.

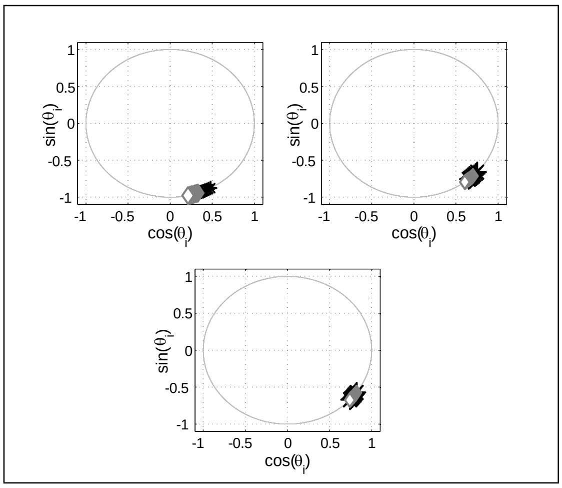

Fig. 7. Complete Model symmetrical bilateral coupling: Phase Plane for $\Delta\omega = 0.1$; a) $k = 0.1$; b)$k = 0.5$; c)$k = 10$

Comparing the plots in Fig. 6 with those in Fig. 7 it can be seen that for the same coupling strength the complete model reaches a smaller $\Delta\theta$, from this it may be implied that for a given $\Delta\theta$ the coupling strength needed in the complete model is smaller than that for the phase model. That is:

$$k_{ph} > k_c \tag{15}$$

being this a direct consequence of the loss of information when passing from the complete two dimensional model to the one dimensional phase model.

5.1 Analysis of Synchronization with the phase model

In order to show why synchronization takes place in the previous section the following analysis is done.

Being the general form of the dynamics of the phase:

$$\dot\theta_1 = m_{1,i}(t) \quad ; \quad \dot\theta_2 = m_{2,i}(t)$$

$m_{j,i}(t)$ are T_j-periodic.

Then both oscillators are coupled by means of $k(\theta_q - \theta_p)$:

$$\begin{aligned} \dot\theta_1 &= m_{1,i}(t) + k(\theta_2 - \theta_1) &= -k\theta_1 + m_{1,i}(t) + k\theta_2 \\ \dot\theta_2 &= m_{2,i}(t) + k(\theta_1 - \theta_2) &= -k\theta_2 + m_{2,i}(t) + k\theta_1 \end{aligned} \tag{16}$$

Expressing (16) in state variables:

$$\begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} m_{1,i}(t) \\ m_{2,i}(t) \end{pmatrix}$$

Then:

$$\begin{vmatrix} \lambda + k & k \\ k & \lambda + k \end{vmatrix} = \lambda^2 + 2k\lambda = \lambda(\lambda + 2k)$$

This means the coupled system has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = -2k$, $\forall k > 0$.

Expressing (16) in terms of the phase difference and the phase addition, $\theta_- \triangleq \theta_1 - \theta_2$ and $\theta_+ \triangleq \theta_1 + \theta_2$:

$$\begin{aligned} \dot\theta_- &= \dot\theta_1 - \dot\theta_2 = -2k(\theta_1 - \theta_2) + m_{1,i}(t) - m_{2,i}(t) \\ \dot\theta_+ &= \dot\theta_1 + \dot\theta_2 = m_{1,i}(t) + m_{2,i}(t) \end{aligned}$$

This is:

$$\begin{aligned} \dot\theta_- &= -2k\theta_- + m_{1,i}(t) - m_{2,i}(t) \\ \dot\theta_+ &= m_{1,i}(t) + m_{2,i}(t) \end{aligned} \tag{17}$$

Seeking for the solution of (17) we have:

$$\begin{aligned} \lim_{t\to\infty} \theta_-(t) &= \frac{m_{1,i} - m_{2,i}}{2k} = \theta_-^* \\ \theta_-^* &\le \frac{1}{2k} \max_{0 \le t \le \overline{T}} M \end{aligned} \tag{18}$$

where $\overline{T} = \max\{T_1, T_2\}$ and

$$M = \left\{ \begin{array}{c} |m_{1,1}(t) - m_{2,1}(t)|, |m_{1,1}(t) - m_{2,2}(t)|, \\ \ldots, |m_{1,1}(t) - m_{2,i}(t)|, \ldots \\ \ldots, |m_{1,2}(t) - m_{2,i}(t)|, \\ \ldots |m_{1,9}(t) - m_{2,9}(t)| \end{array} \right\},$$

$i = 1, \ldots, 9$

Therefore the lowest coupling strength is determined by:

$$k_{\min} = \frac{\overline{m}}{2\theta_-^*} \tag{19}$$

where $\overline{m} = \max_{0 \le t \le \overline{T}} M$ and corresponds to the greatest difference between the slopes of the oscillators.

For instance, for the example illustrated in Fig. 6 where $\omega_1 = 1$, $\omega_2 = 1.1$, if we choose $\theta^*_- = 0.034$, the corresponding lowest coupling strength to guarantee such phase difference is $k_{\min} = 22.7293$. Recalling from (15), $k_c < k_{ph} = k_{\min}$, then $k_{\min}$ guarantees that the complete model attains a phase difference lower than θ^*_-. This is confirmed by the simulation where the needed strength coupling for $\theta^*_- = 0.034$ is $k_c = 2.725$, and given the $k_{\min}$, the phase difference attained is $\theta_- = 0.003558 \ll \theta^*_-$.

Remark 6. The obtained lower bound for the coupling strength, $k_{\min}$, is a quite conservative bound.

On the other hand, from (19) we can solve for θ^*_- for a given coupling strength, this will be the upper bound of θ_-. Making this calculation for each coupling strength used in the examples illustrated in Fig. 6 and Fig. 7 where $\omega_1 = 1$, $\omega_2 = 1.1$ and $k = \{0.1, 0.5, 10\}$ we have:

$$\theta_- \le \begin{cases} 7.7280 & ; \quad k = 0.1 \\ 1.5456 & ; \quad k = 0.5 \\ 0.0773 & ; \quad k = 10 \end{cases} \qquad (20)$$

$$\overline{m} = 1.5456$$

From the simulations done, we notice that the phase difference for the phase model is below the calculated in (20); and as expected, in the case of the complete model the phase difference was much smaller than the above predicted. The results are the following:

$$\theta_{-,ph,sim} = \begin{cases} 0.9694 & ; \quad k = 0.1 \\ 0.6204 & ; \quad k = 0.5 \\ 0.0772 & ; \quad k = 10 \end{cases}$$

$$\theta_{-,c,sim} = \begin{cases} 0.6226 & ; \quad k = 0.1 \\ 0.2383 & ; \quad k = 0.5 \\ 0.0083 & ; \quad k = 10 \end{cases}$$

6. CONCLUSIONS

We developed a one dimensional phase model for the relaxed case of the two dimensional van der Pol oscillator. The phase model construction is based on the slow and fast transitions the phase goes through during its cycle and which we fit to a piece-wise linear function. The proposed model is fully dependent on the period of oscillation and therefore incorporates the effects the nonlinearity parameter ε and natural frequency $\omega \neq 1$ might have on it.

The exemplified application of the phase model is the synchronization analysis of two symmetrically coupled van der Pol oscillators, we use the proposed phase model to calculate the minimum coupling strength needed for their synchronization, this by means of a maximum phase difference allowed for considering the behavior as synchronization. This coupling strength lower bound guarantees that the complete two dimensional model will reach an even smaller phase difference.

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