Dynamical Radial Control of Nonlinear Systems

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Abstract: This paper is concerned with the stabilisation of a general class of nonlinear systems via the associated angular approach. In this method, the system is converted into two subsystems the so called radial and spherical systems. The spherical system is a nonlinear equation on a sphere and the radial system is a scalar differential equation. A stabilising control can be designed based on the one-dimensional radial system dynamics. The radial control may be continuous or discontinuous depending on the structure of the input map. Whenever the input map of the radial subsystem is zero, the radial control is not accessible. In this paper a method is presented to remove this obstacle. The control is designed by including an extra dynamic to the system. Therefore the new system is an augmented system. The radial auxiliary input map of the augmented system i.e. the original control is the new state. Since it is assumed that the original control is not zero, the auxiliary radial control is definable within the operating region.

1. INTRODUCTION

Extensive research activities have been dedicated to the stabilisation problem of nonlinear systems in recent years leading to the development of different approaches; for a comprehensive account see Isidori [1999]-Astolfi et al. [2002]. In (Ortega et al. [2002]) instead of using the classical adaptive control approach a simple nonlinear PI structure is proposed to achieve the system stability. A higher order sliding mode control scheme for uncertain nonlinear systems is proposed by Laghrouchea et al. [2007]. The control design method benefits from integral sliding mode techniques to ensure the robustness and easy implementation.

Banks [1999] has studied the stabilisation of a general class of nonlinear systems by using the angular form of a differential equation to reduce the problem to the control of the radial component of the state. This representation has also been used for studying the Lyapunov spectrum of a nonlinear system (Colonius et al. [1990, 1996]). In this method, the system is converted into two subsystems. One is a nonlinear equation on a sphere (*spherical system*) and the other is a radial differential equation (*radial system*).

In many cases a simple stabilising control law can be derived based on the one-dimensional radial system dynamics. The radial control can be continuous or discontinuous depending on the structure of the input map. Whenever the input map of the radial subsystem is zero, the radial control is not accessible. Therefore the radial control or control design method should be modified such that the defined control is accessible everywhere within the operating region and also stabilises the system. The prime advantage of this method is that the n-dimensional, ($n > 1$), stabilition problem is replaced by a one-dimensional stabilisation problem. Also, the angular control design is straightforward, systematic and can be applied to an extensive class of nonlinear systems including systems with disturbance and un-modelled dynamics. Moreover it can be combined with other approaches such as optimal (Sangelaji et al. [2003]) and adaptive control. In (Sangelaji et al. [2002]), several methods have been presented to remove the singularity points.

In this paper dynamical radial control techniques are employed to remove this obstacle. In this method, an extra dynamic is included in the system. Therefore the new system is an augmented system. The radial auxiliary input map of the augmented system is the new state (the original control). Since it is assumed that the original control is not zero, the auxiliary radial control is therefore definable in the operating region.

This paper is organised as follows: In Section 2 the associated angular method is explained. In Section 3 dynamical radial control (DRC) design procedure is introduced. In section 4 the DRC approach with SMC method is compared. Both methods are applied to a simple example and the simulation results are presented in Section 5. Finally conclusions are presented in Section 6.

2. ASSOCIATED ANGULAR SYSTEM

Consider the nonlinear system

$$\dot{x} = A(x) + B(x)u \tag{1}$$

where $x = [x_1 \ \dots \ x_n]^T \in \mathbb{R}^n$ is the state, u is the scalar control, $A(x) \in \mathbb{R}^n$ and $B(x) \in \mathbb{R}^n$. Let $\mathbb{S}^n \subseteq \mathbb{R}^n$ be the unit n-ball, i.e. $S^n = \{z \in \mathbb{R}^n : \|z\| = 1\}$ and $\mathbb{R}^+$ be the set of positive real numbers. The map

$$\varphi : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R}^+ \times \mathbb{S}^n$$

$$x \to \left(\|x\|, \frac{x}{\|x\|}\right)$$

is a diffeomorphism from $\mathbb{R}^n - \{0\}$ onto $\mathbb{R}^+ \times \mathbb{S}^n$. Note that even as x tends to zero, $\frac{x}{\|x\|}$ $(= z)$ is on the unit ball. The origin is removed from the domain of the function φ; otherwise the origin corresponds to the infinity pair $(0, z)$ where z is any point in $\mathbb{S}^n$. This obstacle can be removed if a unique pair say $(0, z_0)$ with $z_0 = (1, \dots, 0)$ corresponds to the origin. Using diffeomorphism φ the system (1) is converted into the associated radial and spherical subsystems as presented in the following Lemma.

Lemma 1. The system (1) can be written in the form

$$\dot{r} = \lambda_A + \lambda_B u \tag{2}$$

$$\dot{z} = \frac{1}{r}[\bar{A}(r,z) + \bar{B}(r,z)u] \tag{3}$$

from which the following control is obtained

$$u = -\frac{\lambda_A + \alpha}{\lambda_B} \tag{4}$$

where

$$\bar{A}(r,z) = A - \left(z^T A(r,z) z\right) I$$

$$\bar{B}(r,z) = B - \left(z^T B(r,z) z\right) I$$

Also

$$\lambda_A = z^T A(r,z)$$

$$\lambda_B = z^T B(r,z)$$

and $r = \|x\|$, $z = \frac{x}{\|x\|}$. The design parameter $\alpha > 0$ is a constant real number. Moreover, the control (4) stabilises the system (1).

Proof. Dividing (1) by $\|x\|$

$$\frac{\dot{x}}{\|x\|} = \frac{A(x)}{\|x\|} + \frac{B(x)}{\|x\|}u$$

Let $r = \|x\|$ and $\|x\|^2 = x^T x$ then

$$\frac{d}{dt}\|x\|^2 = 2\|x\|\frac{d\|x\|}{dt}$$

and

$$\frac{d}{dt}(x^T x) = \dot{x}^T x + x^T \dot{x} = 2x^T \dot{x}$$

therefore

$$2\|x\|\frac{d\|x\|}{dt} = 2x^T \dot{x}$$

Thus

$$\|x\|\frac{d\|x\|}{dt} = x^T \left(A(x) + B(x)u\right)$$

and

$$\dot{r} = \frac{d\|x\|}{dt} = \frac{x^T A(x)}{\|x\|} + \frac{x^T B(x)}{\|x\|}u$$

so

$$\dot{r} = \frac{x^T (A(x) + B(x)u)}{r} \tag{5}$$

Substituting $x = rz$ into (5) yields

$$\begin{aligned} \dot{r} &= z^T (A(r,z) + B(r,z)u) \\ &= \lambda_A + \lambda_B u \end{aligned} \tag{6}$$

On the other hand, since $z = \frac{x}{\|x\|}$,

$$\begin{aligned} \dot{z} &= \frac{\dot{x}\|x\| - x\frac{d}{dt}\|x\|}{\|x\|^2} \\ &= \frac{\dot{x}}{\|x\|} - \frac{1}{\|x\|^2}x\frac{d}{dt}\|x\| \\ &= \left(\frac{A(x)}{\|x\|} + \frac{B(x)}{\|x\|}u\right) - \frac{1}{\|x\|^2}\left(\frac{x^T A(x)}{\|x\|} + \frac{x^T B(x)}{\|x\|}u\right)x \end{aligned}$$

Therefore, the spherical subsystem is

$$\begin{aligned} \dot{z} &= \frac{1}{r}\left[\left(A(r,z)z + B(r,z)zu\right) - \left(z^T A(r,z)z + z^T B(r,z)zu\right)\right] \\ &= \frac{1}{r}\left[\left(A(r,z) - (z^T A(r,z)z)I\right) + \left(B(r,z)\right.\right. \\ &\qquad \left.\left. - (z^T B(r,z)z)I\right)u\right] \\ &= \frac{1}{r}\left[\bar{A}(r,z) + \bar{B}(r,z)u\right] \end{aligned}$$

If a control is designed such that the scalar radial variable r tends to zero, the system is then stabilised. Substituting the control (4) into (2) yields

$$\dot{r} = -\alpha r$$

which proves the stability of the subsystem (2) and therefore, the system (1).

Note that the z-subsystem operates on the unit ball and the r-subsystem is scalar. The real positive number α is a design parameter and only affects the degree of the stability of the system. In other words, for large values of α the state settling time is shorter in comparison with small values of α. One way to ensure the accessibility of the control (4) is to consider some specific constrains on α. Another way to remove singularities of a discontinuous radial control is to consider an appropriate approximation of the radial control. The following control is a suitable approximation of the radial control (4)

$$u = \begin{cases} -\dfrac{\lambda_A + \alpha r^2}{\lambda_B} & |\lambda_B| > \varepsilon \\[2ex] -\dfrac{\lambda_A + \alpha r^2}{\varepsilon}\ \mathrm{sgn}(\lambda_B) & |\lambda_B| \le \varepsilon \end{cases} \tag{7}$$

However, a more general approach is considered in Section 3 to remove this obstacle.

3. DYNAMICAL RADIAL CONTROL (DRC)

The radial control design may not be applicable for some nonlinear systems in which the control is not defined for the singular points. In this paper dynamical radial control is considered to design a control which is available for all t and stabilises the system. The resulted system has a higher dimension in comparison with the original system in which the state radial map differs from the original system and the input radial map is $x_{n+1}(= u)$. Consider the system

$$\dot{x} = A(x) + B(x)u$$

Select $u = x_{n+1}$ and $v = \dot{u} = \dot{x}_{n+1}$ then the system is

$$\begin{cases} \dot{x} = A(x) + B(x)x_{n+1} \\ \dot{x}_{n+1} = v \end{cases} \tag{8}$$

Therefore, the system (8) can be converted into

$$\begin{pmatrix} \dot{x} \\ \dot{x}_{n+1} \end{pmatrix} = \begin{pmatrix} A(x) + B(x)x_{n+1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

$$\dot{\tilde{x}} = \tilde{A}(x) + \tilde{B}v$$

where v is an auxiliary control. The system in the associated angular form is

$$\dot{\tilde{r}} = \tilde{z}^T \tilde{A}(rz) + \tilde{z}^T \tilde{B}v$$
$$\dot{\tilde{z}} = \frac{1}{\tilde{r}}\big[(\tilde{A}(rz) - \tilde{z}^T \tilde{A}(rz)\tilde{z}) + (\tilde{B} - \tilde{z}^T \tilde{B}\tilde{z})v\big]$$

where $\tilde{r} = \|\tilde{x}\|$ and

$$\|\tilde{x}\|^2 = \|x\|^2 + x_{n+1}^2 = x^T x + x_{n+1}^2$$

Therefore

$$\tilde{r}\dot{\tilde{r}} = \begin{pmatrix} x^T x_{n+1} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{x}_{n+1} \end{pmatrix}$$

and

$$\dot{\tilde{r}} = \frac{\tilde{x}^T \big(\tilde{A}(x) + \tilde{B}(x)v\big)}{\tilde{r}} \tag{9}$$

Substituting

$$\tilde{x} = \tilde{r}\tilde{z} \Longrightarrow \tilde{z} = \frac{\tilde{x}}{\tilde{r}}$$

into (9) yields

$$\dot{\tilde{z}} = \big[\frac{1}{\tilde{r}}\tilde{A}(x) - \frac{1}{\tilde{r}^3}\tilde{x}^T \tilde{A}\tilde{x}\big] + \big[\frac{1}{\tilde{r}}\tilde{B}(x) - \frac{1}{\tilde{r}^3}\tilde{x}^T \tilde{B}(x)\tilde{x}\big]v$$

So the radial control 9 is now

$$v = -\frac{\lambda_{\tilde{A}} + \alpha\tilde{r}^2}{\lambda_{\tilde{B}}}$$

where

$$\lambda_{\tilde{A}} = \tilde{x}^T \tilde{A} = x^T A(x) + x^T B(x)x_{n+1}$$

and

$$\lambda_{\tilde{B}} = \tilde{x}^T \tilde{B} = x_{n+1}$$

Therefore

$$v = -\frac{x^T A(x) + x^T B(x)x_{n+1} + \alpha(x^T x + x_{n+1}^2)}{x_{n+1}} = -\frac{x^T A(x) + x^T B(x)x_{n+1} + \alpha\|x\|^2}{x_{n+1}} - \alpha x_{n+1}$$

provided that $x_{n+1} \neq 0$. The actual control is obtained from the following equation

$$\dot{u} = -\frac{x^T A(x) + x^T B(x)u + \alpha\|x\|^2 + \alpha u^2}{u}$$

In fact, the actual control law satisfies the following differential equation

$$\dot{u}u + x^T B(x)u + \alpha u^2 + x^T A(x) + \alpha\|x\|^2 = 0$$

or

$$\frac{1}{2}\frac{d(u^2)}{dt} + \alpha u^2 + x^T B(x)u + x^T A(x) + \alpha x^T x = 0$$

This method is applicable to a wide class of nonlinear systems in which there exists points such that $\lambda_B = 0$. Using this method the original control input u is considered as a state ($u = x_{n+1}$) of the augmented system, then its radial input map is nonzero everywhere within the operating region. Note that the virtual (auxiliary) control is $\dot{u}(= v)$.

4. COMPARISON WITH SMC

The classical SMC may be used to control the nonlinear systems, particularly when there is an uncertainty in the system. This technique consists of two stages. First a sliding surface is designed and then a discontinuous control is selected to enforce the trajectories to move onto a sliding surface in a finite time and remain on it thereafter. Although it is claimed that this approach posses robustness properties but the chattering phenomenon is a major obstacle for implementation of SMC in a wide range of applications. In comparison to the angular approach the system trajectories may move within the entire space. The projections of the system trajectories on the unit ball are z-subsystem trajectories. If the order of the system is greater than the number of inputs, the order of the system in a sliding mode without considering any compensator, may be less than or the same as the original system but the order of the radial subsystem in the angular approach is always one. In addition, SMCs are usually discontinuous, while an original radial control may be continuous or discontinuous depending on the nature of the system. Note that a discontinuous controller may not be an SMC if the trajectories do not move onto a specific surface in a finite time and stay on it for future time.

5. EXAMPLE

Consider the system

$$\dot{x} = \begin{pmatrix} -3x_1 \\ -x_1 + x_2 \end{pmatrix} + \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} u \tag{10}$$

$x_{n+1} = u$ and $\dot{x}_{n+1} = v$, then

$$\lambda_{\tilde{A}} = -3x_1^2 - x_1x_2 + x_2^2$$
$$\lambda_{\tilde{B}} = x_{n+1}$$

The radial control is

$$v = -\frac{\lambda_{\tilde{A}} + \alpha(x^T x + x_{n+1}^2)}{\lambda_{\tilde{B}}} \tag{11}$$

Fig. 1 and Fig. 2 show the system response using the DRC method with control (11), $\alpha = 0.3$ and $\alpha = 3$ respectively.

The discontinuous radial control (7) can also be used to remove the singularities for this example

$$u = \begin{cases} -\dfrac{\lambda_{\tilde{A}} + \alpha\tilde{r}^2}{\lambda_{\tilde{B}}} & |\lambda_{\tilde{B}}| > \varepsilon \\ -\dfrac{\lambda_{\tilde{A}} + \alpha\tilde{r}^2}{\varepsilon}\,\mathrm{sgn}(\lambda_{\tilde{B}}) & |\lambda_{\tilde{B}}| \leq \varepsilon \end{cases} \tag{12}$$

Fig. 3 shows the simulation results with control (12), $\alpha = 2$ and $\varepsilon = 0.02$.

The system (10) may be stabilised using other methods such as SMC. Now a SMC is designed for the system (10) and the results are compared with that of the radial method. Select the sliding surface

$$s = cx_1 + x_2 = 0$$

The SMC

$$u_s = \frac{1}{cx_2 - x_1}\left((3c+1)x_1 - x_2 - K\mathrm{sgn}(s)\right) \tag{13}$$

where $K > 0$, enforces the trajectories to move onto the sliding surface in finite time and remain on it because the

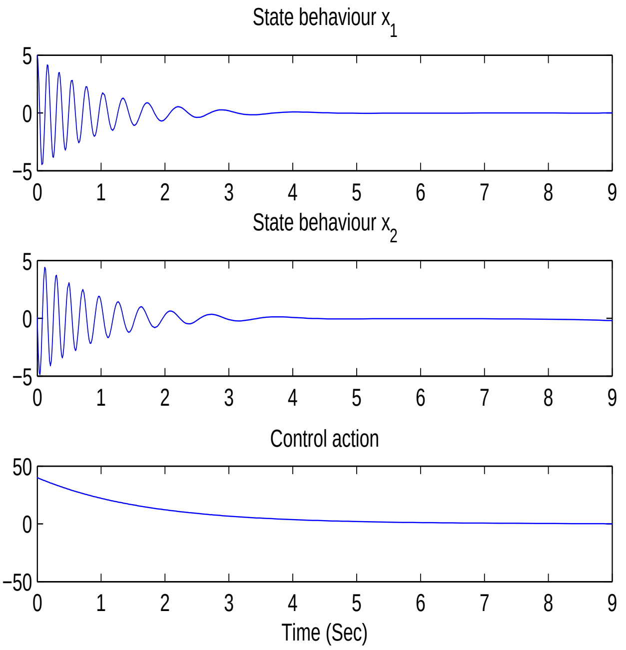

Fig. 1. Simulation results of the DRC (11) with $\alpha = 0.3$

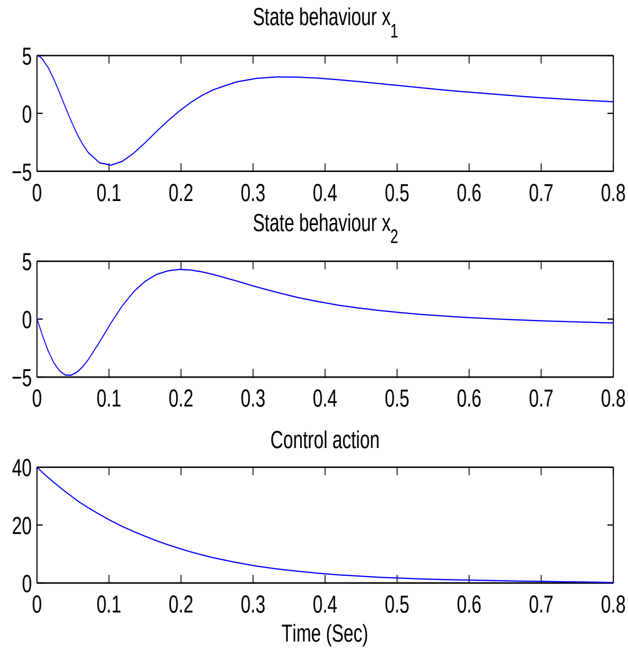

Fig. 2. Simulation results of the DRC with $\alpha = 3$

reaching condition $s\dot{s} < 0$ is satisfied. The radial control needs less effort than the SMC and the settling time using the radial control is shorter than the SMC. See Fig. 4.

The drawback of the SMC (13) is chattering resulting from switching between two different values while the radial controllers (11) and (7) are smooth. However, one may consider the continuous approximation of the SMC or design a higher-order SMC for the system to eliminate the chattering and compare the designed SMCs with the radial counterparts. One way to remove the chattering is to

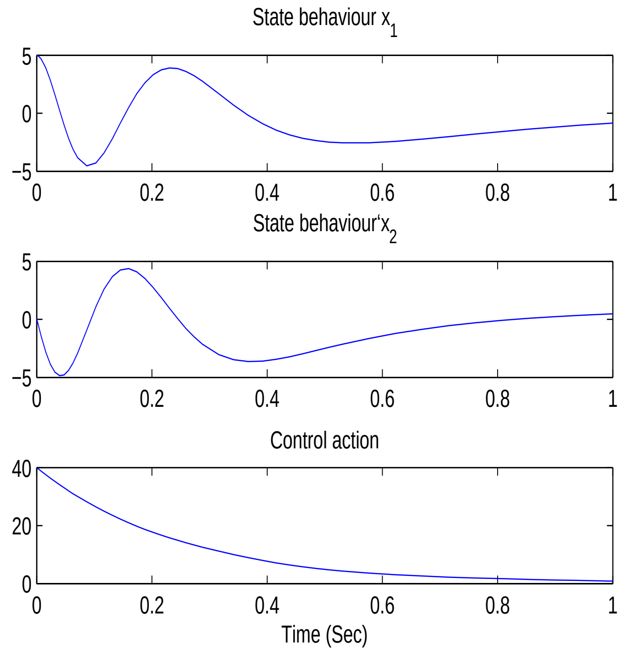

Fig. 3. Simulation results of the discontinuous DRC (7) with $\alpha = 2$

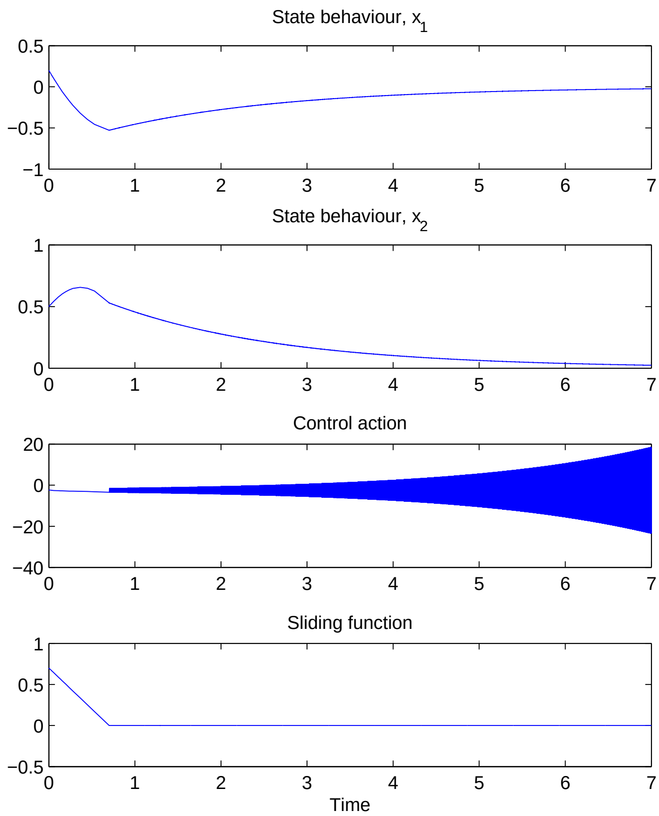

Fig. 4. Responses of example with the SMC (13)

consider the continuous approximation of the SMC (13). Consider the control

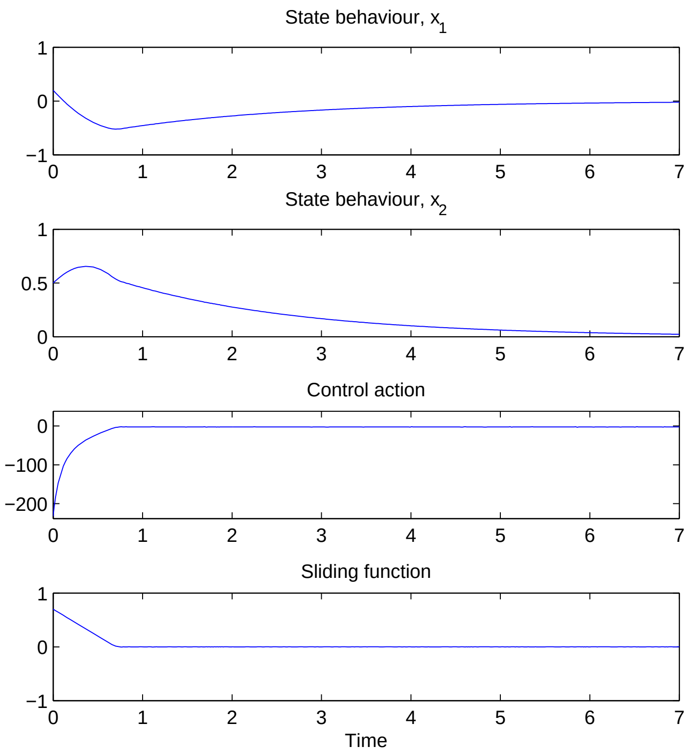

Fig. 5. Responses of example with the control (14)

$$u_s = \frac{1}{cx_2 - x_1}\left((3c+1)x_1 - x_2\right) - \frac{K}{cx_2 - x_1} sat(s) \quad (14)$$

where

$$sat(s) = \begin{cases} 1 & \text{if} \quad s > \eta \\ \frac{s}{\eta} & \text{if} \quad |s| \leq \eta \\ -1 & \text{if} \quad s < -\eta \end{cases}$$

and $\eta > 0$ is a small real number. This approximation of the SMC guarantees the system trajectories lie within a boundary layer of the sliding surface. Although the approximation SMC (14) removes the chattering problem but it does not yield a shorter state settling time. Also less control effort is needed with the radial controllers. Fig. 5 shows the simulation result using the control (14) with $\eta = 0.1$.

6. CONCLUSION

The control design and stabilisation problem of nonlinear control systems based on the associated angular method have been studied. Dynamical radial control is developed to design a suitable radial auxiliary control to remove the problem of singularity points. The proposed approach is capable of removing singular points and in some cases reducing them. In the latter a discontinuous radial control may be used. A comparison of DRC and SMC is given and an example has been used to illustrate the applicability of both methods. It is shown that the DRC method achieves better results.

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