A non-parametric method for non-linear J-J′ spectral factorisation.

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Abstract:
This paper presents a Newton-iteration method for obtaining a J-J′ spectral factorisation of systems from non-parametric characterisations using identification techniques. Systems are assumed to be Fréchet differentiable discrete-time maps. The technique may be used on nonlinear nonparametric time-response representations. The scheme requires stabilised identification and stabilised inverse identification and with such can be used to iterate on NARMAX controller structures.

Keywords: H_∞, J-spectral factorisation, NARMAX, Newton iterations, Nonlinear, Robust control.

1. INTRODUCTION

Non parametric controller design methods can capture the behaviour of an experimental system without the significant distortions due to unnecessary assumptions about model structure. They allow the ability to fix the structure of the controller, e.g. for implementing in engine-management system (EMS) software, without this being imposed by assumptions on the model structure. Such methods are currently limited largely to linear rational and linear irrational systems. J-spectral factorisation are well known to offer solutions to general H_∞ control problems in both linear [8] and nonlinear cases [**?**]. The purpose of this paper is to present an identification based nonparametric technique for obtaining J-spectral factorisations of discrete-time smooth nonlinear systems. The technique is thus applicable to linear and nonlinear discrete systems.

2. ASSUMPTIONS AND NOTATION

The input-output relation of [17] is conveniently used interchangeably with the operator form of a map. For partitioned inputs and outputs, we adopt the notation of [12] where the solutions of a map $G^{w,z}_{u,y}$ on $l_2^{n_u} \times l_2^{n_y} \times l_2^{n_w} \times l_2^{n_z}$ are written in relational form $(u, y, w, z) \in G^{w,z}_{u,y}$. The first elements u, y are the input vector valued signal sequences and the second w, z are the output sequences. We then interchangeably also use the operator form $G^{w,z}_{u,y}$: $l_2^{n_u} \times l_2^{n_y} \mapsto l_2^{n_w} \times l_2^{n_z}$ and $\begin{smallmatrix} w \\ z \end{smallmatrix} = G^{w,z}_{u,y} \begin{smallmatrix} u \\ y \end{smallmatrix}$. All maps are assumed Fréchet differentiable and we denote the Fréchet derivative of $G^{w,z}_{u,y}$ by $DG^{w,z}_{u,y}$. We will take inverse stable as equivalent to minimum-phase (MP). Denote the signature matrices $J = J_{n_w,n_z}$ and $J' = J_{n_{\bar{w}},n_{\bar{z}}}$.

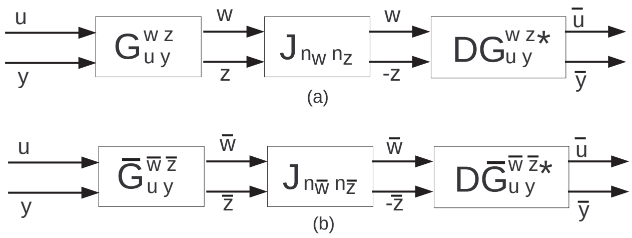

Fig. 1. J-J′ Equivalent System

3. J-SPECTRAL FACTORISATION AND J-J′ EQUIVALENCE

The central problem addressed in this paper is (see Fig, (1)), given a map $G^{w,z}_{u,y}$, find an invertible stable, MP map $\bar{G}^{\bar{w},\bar{z}}_{u,y}$ so that the equality

$$||z||_2^2 - ||w||_2^2 = ||\bar{z}||_2^2 - ||\bar{w}||_2^2 \tag{1}$$

is satisfied on the maps $G^{w,z}_{u,y}$ and $\bar{G}^{\bar{w},\bar{z}}_{u,y}$ for $\forall(\bar{w}, \bar{z}) \in l_2^{n_{\bar{w}}} \times l_2^{n_{\bar{z}}}$ and $\forall(u, y) \in l_2^{n_u} \times l_2^{n_y}$. This is an equivalent formulation of the inner-outer J-J′ spectral factorisation problem.

Šebek [11] gives a thorough account of solving J-Spectral Factorisation problems in the linear matrix polynomial (ie $D\bar{G}^{\bar{w},\bar{z}}_{u,y} = \bar{G}^{\bar{w},\bar{z}}_{u,y}$) case. The methods employed are 1/ diagonalisation 2/ successive factor extraction 3/ interpolation and 4/ solution of the algebraic Riccati equation.

The method for stable and MP solutions to the J-Spectral Factorisation problem in the linear state-space case [4],[8] is to first to find a stable solution in a (J-J) factorisation and then to find the required MP (J-J′) solution using this stable plant. An anti-stabilising right J-lossless conjugator θ_+ of $G^{w,z*}_{u,y}J$ is found by solution of a first Riccati equation to give the stable plant $J\theta_+^* JG^{w,z}_{u,y}$. Solution of a second Riccati equation then gives the MP (J-J′) factorisation of this plant whilst preserving its stability.

Advanced nonlinear treatments are considered in [10] [5] (and in the extensive references therein) approached by solving Hamilton-Jacobi Bellman equations which are the nonlinear extension of the Riccati approach. Nonlinear chain-scattering approaches are given in [2], [6].

The alternative approach here is to seek a square invertible stable MP J-J′ unitary plant $\bar{G}^{\bar{w},\bar{z}}_{u,y}$ in parametric format (with inverse $\bar{G}^{u,y}_{\bar{w},\bar{z}}$) equivalent to the non-square system $G^{w,z}_{u,y}$ described in non-parametric format. For application in H_∞ control the partial inverse $\theta^{w,\bar{z}}_{\bar{w},z}$ of $\theta^{w,z}_{\bar{w},\bar{z}} = G^{w,z}_{u,y}\bar{G}^{u,y}_{\bar{w},\bar{z}}$ is then further required to be stable, so that $\theta^{w,\bar{z}}_{\bar{w},z}$ is also J-J′ lossless.

4. ON INTEGRATING FRÉCHET DIFFERENTIABLE MAPS

Let $T : U \mapsto Y$, where U, Y are *any* normed vector spaces. Write $y = T(u)$, $u \in U$ and $y \in Y$. If $\int_0^1 T(\lambda u_1)d\lambda$ exists it is called the direct (Riemann) integral and is written $\int_0^{u_1} T(u)du$. The analogue of the *fundamental theorem of calculus* is then

$$y_1 = T(u_1) = \int_0^{u_1} DT(u)(u_1)du = \int_0^1 DT(\lambda u_1)(u_1)d\lambda$$

Let $T(u) = \langle J\Phi(u)|\Phi(u)\rangle$ then

$$DT(\lambda u_1)(u_1) = 2\,\langle J\Phi(\lambda u_1)|D\Phi(\lambda u_1)(u_1)\rangle$$

and

$$y_1 = T(u_1) = \int_0^1 DT(\lambda u_1)(u_1)d\lambda$$

$$= 2\int_0^1 \langle J\Phi(\lambda u_1)|D\Phi(\lambda u_1)(u_1)\rangle\, d\lambda$$

which is conventionally [1] written

$$2\int_0^1 \langle J\Phi(u)|D\Phi(u)(u_1)\rangle\, du$$

5. MAIN THEOREMS

Two theorems, originally presented in [12] are required for the Newton iteration scheme:

Theorem 5.1 ◁ *For Fréchet differentiable maps* $\Phi : U \to Y$ *and* $\Phi' : U \to Y'$ *where* U, Y, Y' *are any normed vector spaces the equality*

$$\langle J\Phi(u)|\Phi(u)\rangle = \langle J'\Phi'(u)|\Phi'(u)\rangle \tag{2}$$

holds $\forall u \in U$ *iff*

$$[D\Phi(u)]^* J\Phi(u) = [D\Phi'(u)]^* J'\Phi'(u) \tag{3}$$

holds $\forall u \in U$.

Proof: If 2 holds then differentiating with respect to u in the direction of any $u_1 \in U$ gives

$$2\,\langle J\Phi(u)|D\Phi(u)(u_1)\rangle = 2\,\langle J'\Phi'(u)|D\Phi(u)(u_1)\rangle$$

and so

$$\langle [D\Phi(u)]^* J\Phi(u)|u_1\rangle = \langle [D\Phi'(u)]^* J'\Phi'(u)|u_1\rangle$$

which gives 3.

If 3 holds then

$$\langle [D\Phi(\lambda u_1)]^* J\Phi(\lambda u_1)|u_1\rangle = \langle [D\Phi'(\lambda u_1)]^* J'\Phi'(\lambda u_1)|u_1\rangle$$

and so

$$\langle J\Phi(\lambda u_1)|D\Phi(\lambda u_1)(u_1)\rangle = \langle J'\Phi'(\lambda u_1)|D\Phi'(\lambda u_1)(u_1)\rangle$$

both holding for all $\lambda \in \mathbb{R}$ and $u_1 \in U$. Integrating we obtain

$$\int_0^1 \langle J\Phi(\lambda u_1)|D\Phi(\lambda u_1)(u_1)\rangle\, d\lambda =$$

$$\int_0^1 \langle J'\Phi'(\lambda u_1)|D\Phi'(\lambda u_1)(u_1)\rangle\, d\lambda$$

which we identify with

$$\frac{1}{2}\int_0^1 D\,\langle J\Phi(\lambda u_1)|\Phi(\lambda u_1)\rangle\,(u_1)d\lambda =$$

$$\frac{1}{2}\int_0^1 D\,\langle J'\Phi'(\lambda u_1)|\Phi'(\lambda u_1)\rangle\,(u_1)d\lambda$$

and thus we obtain

$$\langle J\Phi(u_1)|\Phi(u_1)\rangle = \langle J'\Phi'(u_1)|\Phi'(u_1)\rangle\,.$$

▷. (See Theorem 1 of [1] for a related result restricted to $\mathcal{L}_2$)

Theorem 5.2 ◁ *The equality*

$$||z||_2^2 - ||w||_2^2 = ||\bar{z}||_2^2 - ||\bar{w}||_2^2 \tag{4}$$

is satisfied on the maps $G^{w,z}_{u,y}$ *and* $\bar{G}^{\bar{w},\bar{z}}_{u,y}$ *for* $\forall(\bar{w},\bar{z}) \in l_2^{n_w'} \times l_2^{n_{z'}}$ *and* $\forall(u,y) \in l_2^{n_u} \times l_2^{n_y}$ *if*

$$[DG^{w,z}_{u,y}]^* J_{n_w,n_z} G^{w,z}_{u,y} = [D\bar{G}^{\bar{w},\bar{z}}_{u,y}]^* J_{n_{w'},n_{z'}} \bar{G}^{\bar{w},\bar{z}}_{u,y} \tag{5}$$

where $\bar{G}^{\bar{w},\bar{z}}_{u,y}$ *is invertible and stable.* **Proof:** Follows directly from Theorem 5.1 and the invertibility of $\bar{G}^{\bar{w},\bar{z}}_{u,y}$. ▷.

6. NUMERICAL METHODS

6.1 Newton Iteration

Our Newton-Raphson iteration scheme is a generalisation of the matrical polynomial I-spectral factorisation method due to Tunnicliffe-Wilson [13][14] and Vostry [15][16] (see also [9]).

We require a Newton iteration on $F = D\Phi^* J\Phi - DB^* JB$. The iteration scheme is then $\Phi_{i+1} = \Phi_i + DF_i^{-1}F(\Phi_i) = 0$ or $F(\Phi_i) + DF_i(\Phi_i - \Phi_{i+1}) = 0$, written $F_i + \delta F_i = 0$. [2] Let $\Phi_{i+1} = \Phi_i + D\Phi_i$, then from $\delta F_i = D^2\Phi_i^* J\Phi_i + D\Phi_i^* JD\Phi_i$, we have $D\Phi_{i+1}^* J\Phi_i + D\Phi_i^* J\Phi_{i+1} = D[\Phi_i + D\Phi_i]^* J\Phi_i + D\Phi_i^* J[\Phi_i + D\Phi_i] = \{D^2\Phi_i^* J\Phi_i + D\Phi_i^* JD\Phi_i\} + 2D\Phi_i^* J\Phi_i = \delta F_i + 2D\Phi_i^* J\Phi_i$.

From above this implies the iteration scheme

[1] and to the author confusingly!

[2] The Fréchet derivative, DF_i is a linear map but the Fréchet differential, δF_i is in general a nonlinear map.

$$D\Phi_{i+1}^* J\Phi_i + D\Phi_i^* J\Phi_{i+1} - D\Phi_i^* J\Phi_i - DB^* JB = 0 \quad (6)$$

Setting $X_i = \Phi_i + 2D\Phi_i$ (i.e. $\Phi_{i+1} = \Phi_i + D\Phi_i = \frac{1}{2}(\Phi_i + X_i)$) then the scheme is $2(F_i + \delta F_i) = D[\Phi_i + X_i]^* JX_i + D\Phi_i^* J[\Phi_i + X_i] - 2D\Phi_i^* J\Phi_i - 2DB^* JB = 0$, that is

$$2(F_i + \delta F_i) = 2DB^* JB - DX_i^* J\Phi_i - D\Phi_i^* JX_i = 0 \quad (7)$$

The solution sequence $\Phi_0, \Phi_1, \ldots$ for (7) is then subject to (6). This approach is used later to suggest a Quasi-Newton scheme.

6.2 Time Domain Characterisation of the Adjoint System

For any non-negative indexed sequence [3] $u = \{u_0, u_1, u_2, \ldots\}$ denote as u^R the negative (*reversed*) sequence $u^R = \{\ldots, x_{-2}, x_{-1}, x_0\}$ where $x_0 = u_0, x_{-1} = u_1, \ldots$. For any input-output map $(u, w, y, z) \in G_{u,w}^{y,z}$ we will denote as $G_{u,w}^{y,z\,R}$ the reverse time map $G_{u,w}^{y,z\,R} = \{(u^R, w^R, y^R, z^R) : (u, w, y, z) \in G_{u,w}^{y,z}\}$.

For linear time-invariant systems the transpose of the above *reverse-time map* is the adjoint of the map, $(G_{u,w}^{y,z\,R})^T = G_{u,w}^{y,z\,*}$. For nonlinear systems the appropriate adjoint system is the adjoint of the Fréchet derivative (see also [7], [1] [5]). In the proposed Newton iteration reverse time simulation is useful since if the Fréchet derivative is stable its adjoint is antistable and it is then preferable to simulate with the stable transpose.

6.3 Iterative J-Spectral Factorisation Scheme

Let $\bar{G}_{u,y}^{\bar{w},\bar{z}}$ be a stable MP solution of

$$[DG_{u,y}^{w,z}]^* J_{n_w,n_z} G_{u,y}^{w,z} = [D\bar{G}_{u,y}^{\bar{w},\bar{z}}]^* J_{n_{w'},n_{z'}} \bar{G}_{u,y}^{\bar{w},\bar{z}} \qquad (8)$$

valid for $\forall (u, y) \in l_2^{n_u} \times l_2^{n_y}$. To solve (8) we set up the iteration scheme: define the nonlinear maps $\bar{G}_{u,y}^{\bar{w},\bar{z}}[j]$ and $X_{u,y}^{\bar{w},\bar{z}}[j]$ by the recursions with the damping parameter λ, $0 < \lambda < 1$:

$$\begin{aligned} D\bar{G}_{u,y}^{\bar{w},\bar{z}}[j]^* J_{n_w,n_z} X_{u,y}^{\bar{w},\bar{z}}[j] + DX_{u,y}^{\bar{w},\bar{z}}[j]^* J_{n'_w,n'_z} \bar{G}_{u,y}^{\bar{w},\bar{z}}[j] \\ = 2D\bar{G}_{u,y}^{\bar{w},\bar{z}}[j]^* J_{n_w,n_z} \bar{G}_{u,y}^{\bar{w},\bar{z}} \qquad (9) \\ \bar{G}_{u,y}^{\bar{w},\bar{z}}[j+1] = (1-\lambda)\bar{G}_{u,y}^{\bar{w},\bar{z}}[j] + \lambda X_{u,y}^{\bar{w},\bar{z}}[j] \end{aligned}$$

starting with a stable MP map $\bar{G}_{u,y}^{\bar{w},\bar{z}}[0]$. Then $\bar{G}_{u,y}^{\bar{w},\bar{z}} = \lim_{j\to\infty} \bar{G}_{u,y}^{\bar{w},\bar{z}}[j]$ The solution for $X_{u,y}^{\bar{w},\bar{z}}[j]$ to (9) is is obtainable as the solution of a simple linear equation in the linear rational polynomial case. In the nonlinear case it is itself solved by the secondary Newton iteration: define map $P_{u,y}^{\bar{w},\bar{z}}[i]$ and $Q_{u,y}^{\bar{w},\bar{z}}[i]$ where $P_{u,y}^{\bar{w},\bar{z}}[i]$ has the structure of $DQ_{u,y}^{\bar{w},\bar{z}}[i]$, by the recursions

$$\begin{aligned} P_{u,y}^{\bar{w},\bar{z}}[i]^* J_{n'_w,n'_z} \bar{G}_{u,y}^{\bar{w},\bar{z}}[j] + D\bar{G}_{u,y}^{\bar{w},\bar{z}}[j]^* J_{n_w,n_z} Q_{u,y}^{\bar{w},\bar{z}}[i] \\ = 2D\bar{G}_{u,y}^{\bar{w},\bar{z}*} J_{n_w,n_z} \bar{G}_{u,y}^{\bar{w},\bar{z}} \\ P_{u,y}^{\bar{w},\bar{z}}[i+1] = (1-\lambda)P_{u,y}^{\bar{w},\bar{z}}[i] + \lambda Q_{u,y}^{\bar{w},\bar{z}}[i] \end{aligned}$$

starting with a stable MP map $P_{u,y}^{\bar{w},\bar{z}}[0]$. Then $X_{u,y}^{\bar{w},\bar{z}}[j] = \lim_{j\to\infty} Q_{u,y}^{\bar{w},\bar{z}}[i]$

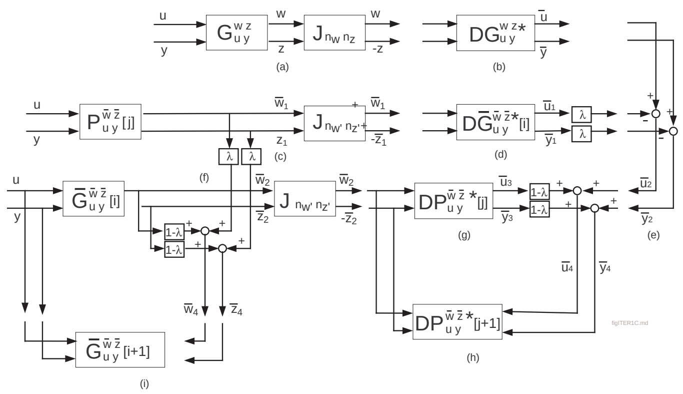

Fig. 2. Numerical Simulations for Newton Iteration

6.4 A Newton Iteration Scheme

To implement the Newton iteration of section 6.3 we adopt the numerical simulations of Fig. 2. We assume that the non-parametric characterisation of the plant, in the form of the adjointed data $(u, y, \bar{u}, \bar{y})$ of Fig. 2, is available. If it is not then the data may be generated for a stable model by the process in the algorithm:

(1) Firstly generate rich [3] input data (u, y). Then determine $(u, y, w, z) \in G_{u,y}^{w,z}$ as in Fig. 2a.
(2) Determine $(\bar{u}, \bar{y})$ either from the simulation $(w, -z, \bar{u}, \bar{y}) \in DG_{u,y}^{w,z*}$ or from the reverse-time simulation $(w^R, -z^R, \bar{u}_r, \bar{y}_r) \in DG_{u,y}^{w,z\,T}$ and setting $(\bar{u}, \bar{y}) = (\bar{u}_r^R, \bar{y}_r^R)$ as in Fig. 2b.

Using the same rich input data (u, y) as in 1 above, the iteration scheme is then:

(1) Start with a stable MP $\bar{G}_{u,y}^{\bar{w},\bar{z}}[j]$ for $j = 0$ and a stable MP $P_{u,y}^{\bar{w},\bar{z}}[i]$ for $i = 0$. Set $j = 0$. Set $i = 0$.
(2) Obtain the the data set $(\bar{w}_2, -\bar{z}_2, u, y)$ from the simulation $(u, y, \bar{w}, \bar{z}) \in \bar{G}_{u,y}^{\bar{w},\bar{z}}[j]$ of Fig. 2f.
(3) Obtain the data set $(u, y, w_1, z_1) \in P_{u,y}^{\bar{w},\bar{z}}[i]$ by simulation as in Fig. 2c.
(4) With $D\bar{G}_{u,y}^{\bar{w},\bar{z}}[j]$ obtain the data set $(\bar{u}_1, \bar{y}_1)$ from the reverse-time simulation $({w_1}^R, -{z_1}^R, \bar{u}_{r1}, \bar{y}_{r1}) \in D\bar{G}_{u,y}^{\bar{w},\bar{z}T}[j]$ and then by setting $(\bar{u}_1, \bar{y}_1) = (\bar{u}_{r1}^R, \bar{y}_{r1}^R)$ as in Fig. 2d.
(5) Compute $\bar{u}_2 = \bar{u} - \lambda\bar{u}_1$ & $\bar{y}_2 = \bar{y} - \lambda\bar{y}_1$ as Fig. 2e. [4]
(6) Determine $(\bar{u}_3, \bar{y}_3)$ from the simulation $(w_2, -z_2, \bar{u}_3, \bar{y}_3) \in DP_{u,y}^{\bar{w},\bar{z}*}[i]$ as in Fig. 2g.
(7) Compute $\bar{u}_4 = \bar{u}_2 + (1-\lambda)\bar{u}_3$ and $\bar{y}_4 = \bar{y}_2 + (1-\lambda)\bar{y}_3$
(8) With the data $(w_2, -z_2, \bar{u}_4, \bar{y}_4)$ identify a stable MP approximation to give $DP_{u,y}^{\bar{w},\bar{z}*}[i+1]$
(9) Increment i and iterate from step 3 until convergence of $P_{u,y}^{\bar{w},\bar{z}}[i]$ to $P_{u,y}^{\bar{w},\bar{z}}[i+1]$ [5]
(10) Compute $w_4 = \lambda w_1 + (1-\lambda)w_2$ and $z_4 = \lambda z_1 + \lambda z_2$.
(11) With the data (u, y, w_4, z_4) identify a stable MP approximation to give $\bar{G}_{u,y}^{\bar{w},\bar{z}}[j+1]$
(12) Set $P_{u,y}^{\bar{w},\bar{z}}[0] = \bar{G}_{u,y}^{\bar{w},\bar{z}}[j+1]$, increment j, set $i = 0$ and and iterate from step 2 until convergence of $\bar{G}_{u,y}^{\bar{w},\bar{z}}[j]$ to $\bar{G}_{u,y}^{\bar{w},\bar{z}}[j+1]$ to obtain $\bar{G}_{u,y}^{\bar{w},\bar{z}} = \lim_{j\to\infty} \bar{G}_{u,y}^{\bar{w},\bar{z}}[j]$.

[3] we will only apply to finite sequences i.e. sequences $u_i = 0; a.e.$

[4] At this stage $Q_{u,y}^{\bar{w},\bar{z}*}[i]$ is characterised by $(\bar{w}_2, -\bar{z}_2, \bar{u}_2, \bar{y}_2)$

[5] At this stage $X_{u,y}^{\bar{w},\bar{z}}[j]$ is characterised by $(u, y, \bar{w}_1, -\bar{z}_1)$

6.5 Preliminary Results

Preliminary numerical results using ordinary least-squares identification with a stabilising/inverse-stabilising regularisation technique applied on affine parameter systems, indicate reliable convergence with small λ parameters. There appears to be little merit in large inner-loop iterations, $i >> 1$ and further, convergence at step 9 does not appear to be necessary. Effective values of λ which allow accurate convergence and maintenance of the stability constraints seem to be $\lambda \approx 0.15$. Update equations using parameter addition rather than identified update can be used with affine parameterised structures, and in this case these are found to be more numerically robust.

In general the J-J$'$ factorisation may be expected to have multiple solutions and corresponding to this, significantly different solutions are obtained with different initial conditions. It is proposed that a search on initial conditions can be used to determine a solution which results in the J-J$'$ unitary inner also having the J-J$'$ lossless stability property required for control. Such solutions have been found in preliminary numerical trials.

Incorporating integral weighting function for integral control action results in a a near marginally stable original plant which is progressively more difficult to factorise as the integral effect is increased. The very slow (low-frequency) behaviour is increasingly more difficult to identify over the simulation time. This parallels the problems of solving the Riccati equations in the linear case when the plant has poles near the marginal stability boundary.

7. CONCLUSIONS

A Newton-iteration scheme is presented for discrete-time Fréchet differentiable maps which determines a J-J$'$ spectral factorisation from non-parametric data.

A recently presented theorem giving an equivalence between a J-J$'$ spectral factorisation equation in the Fréchet derivatives of the original and factorised plant and the J-J$'$ input-output power balance between the original and factorised system is used to develop the scheme.

Identification techniques are used in the iteration. The use of a simultaneous stabilised identification and stabilised inverse identification is required.

The method should allow the development of fixed structure and low order $\mathcal{H}_\infty$ controllers for linear irrational and smooth nonlinear plants.

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