Control Based on Energy for Vertical 2 Link Underactuated Robots.

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Abstract: A class of nonlinear control scheme for swinging up and stabilization of underactuated 2-link robots is introduced. To this end, the control law proposed is applied to an benchmark system. The proposed methodology is designed based on Euler-Lagrange dynamics, energy analysis and Lyapunov theory. A class of linear control doesn't allow to compensate the no linear dynamics performance, for example, inertia, Coriolis, gravity and tribology forces, specially when the system present the underactuated property. The controller in this paper has the advantage stability injection, moreover, we provide a passivity based on stability analysis which suggest that the system has a condition of strictly semi-definite positive realness of tracking energy error and desired position, this is a necessary condition for a stability. Swinging control is based on an energy approach and the passivity properties, and then some conditions on the parameters in the control law such that the total energy of the underactuated robot converges to the potential energy of its top upright position are given. The stabilization system is based on switching LQR control.

1. INTRODUCTION

Underactuated robots are well known benchmark systems Fantoni [2002], where we can mainly investigate the regulation set-point, including swinging up and balancing, as well as trajectories tracking. The underactuated robots have been studied as a typical examples of underactuated mechanical systems, see e.g. Ordaz [2007], Fantoni [2002], Spong [1994].

Nonlinear dynamics with motion constraints and rapidly changing operating conditions, sometimes make such control problems involves. The rigid body mechanics of flight control or robot manipulator motion is often formulated with the general equation obtained from Lagrangian mechanics:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{1}$$

The position coordinates $q \in R^n$ with associated velocities $\dot{q}$ and accelerations $\ddot{q}$ are controlled with the driving forces $\tau \in R^n$. The generalized moment of inertia $D(q)$, the coriolis, centripetal forces $C(q,\dot{q})\dot{q}$, and the gravitational forces $G(q)$ all vary along the trajectories.

The swing up control problem for the underactuated robot is to swing the underactuated robots up to its upright position (top unstable equilibrium position) and balance it about the vertical.

For the swinging up control, Block and Spong [1994] used partial feedback linearization techniques and for the balancing and stabilizing controller, he used linearization about the desired equilibrium point by LQR. Nevertheless, he do not present a stability analysis. The author used concepts such as partial feedback linearization, zero dynamics, and relative degree.

By combining Lyapunov theory with passivity properties and energy shaping, a nonlinear control for some underactuated systems have been designed by Fantoni and Lozano Fantoni [2002], where Lyapunov theory takes an important role in controller design and system convergence analysis. This methodology has been tested on many typical underactuated systems such that Acrobot Spong [1994], Pendubot Xiao [2002], rotating pendulum, cart and pole system, inertial wheel pendulum Fantoni [2002], Zhong [2001], and other underactuated systems.

A drawback in those approaches is that the control law proposed (only to swinging up the pendubot), enters the dynamics of the system into a homoclinic orbit, difficulting to switch on the linear controller and stabilize around the unstable equilibrium point.

2. EULER-LAGRANGE MODEL PROPERTIES AND CONTROL.

The dynamic Euler-Lagrange equations are:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q_i} = \tau_i, \tag{2}$$

From (2) we obtain the generalized Euler-lagrange equation for manipulator robots as (1). Form the Euler-Lagrange formulation one can obtain the mathematical model as:

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ D(q)^{-1}\left[\tau - C(q,\dot{q})\dot{q} - G(q)\right] \end{bmatrix}. \quad (3)$$

2.1 Properties of Euler-Lagrange systems.

The dynamic equation for manipulator robots (2) have the following interesting properties. The inertial matrix $D(q)$ is a $n \times n$ matrix positive definite, these where the entires are only function of q. The Coriolis and centripetal matrix forces $C(q,\dot{q})$ is a $n \times n$ matrix where this elements are function of q and $\dot{q}$, and $C(q,\dot{q})$. $D(q)$ and $C(q,\dot{q})$ have the following properties Kelly [2003]:

- The Euler-Lagrange system have the total energy as:

$$\mathcal{E} = \mathcal{K} + \mathcal{U} \quad (4)$$

By differentiating (4) we obtain

$$\begin{aligned} \dot{\mathcal{E}} &= \dot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T G(q) \\ &= \dot{q}^T \tau. \end{aligned} \quad (5)$$

- From the passivity property we have that:

$$V(x) - V(x_0) \leq \int_0^t y^T(s)u(s)ds, \quad (6)$$

where $V(x)$ is a storage function, $y(s)$ is the output, and $u(s)$ is the input of the system. Using (5) for the Euler-Lagrange system, energy function $\mathcal{E}$ as the storage function, and we have the passivity property as:

$$\mathcal{E}(t) - \mathcal{E}(0) \leq \int_0^t \dot{q}^T \tau dt \quad (7)$$

where $\dot{q}$ is the output, and τ is the input of the system.

These Euler-Lagrange properties are used in the stability control analysis.

3. ENERGY BASED CONTROL FOR VERTICAL 2 LINK UNDERACTUATED ROBOT.

Inspired by control approach Ordaz and Domínguez 2 [2007], we can give an extension of this approach to the vertical two link underactuated robot. When the degree of freedom $n = 2$, and the robot has the underactuated property, then the rank of input forces $rank(\tau) < \dim(q)$. For two link, $\tau \in R^{2\times 1}$ and $q \in R^{2\times 1}$, then the control law Ordaz and Domínguez 2 [2007] can be written as:

$$\tau_i = -\frac{1}{kd_{ki}}\left(\dot{q}_i + kp_i\tilde{q}_i\right) + g_i(q) + F_i(\dot{q}), \quad (8)$$

where $i = 1$ or $i = 2$. It depend son to actuated link, and the condition for swinging is given by $\tilde{q}_i$ Ordaz and Domínguez [2007]. By using (8) if the error function $\tilde{q}_i = 0$ and the non-actuated link is initialized at an other, by dissipative forces property the system dynamic remain in the equilibrium point.

3.1 Underactuated dynamical system.

Generally the dynamical model of an underactuated system can be stated as follow:

$$D(q)(\ddot{q}_a, \ddot{q}_u) + C(q,\dot{q})(\dot{q}_a, \dot{q}_u) + G(q) = \tau, \quad (9)$$

where subindex a indicate the actuated link, and subindex u indicate the underactuated link. For 2-DOF underactuated robot the math model can be written as follows:

$$\begin{bmatrix} \ddot{q}_a \\ \ddot{q}_u \end{bmatrix} = D(q)^{-1}\left(\begin{bmatrix} \tau \\ 0 \end{bmatrix} - C(q,\dot{q})\begin{bmatrix} \dot{q}_a \\ \dot{q}_u \end{bmatrix} - G(q)\right), \quad (10)$$

then the coordinate actuated can be write as follow:

$$\ddot{q}_a = \frac{f(q_a, q_u, \dot{q}_a, \dot{q}_u) + g(q)\tau}{\Delta}, \quad (11)$$

where $\Delta = \det(D(q))$.

3.2 Control law synthesis.

In order to stated our control approach, we need the total energy. The system total energy, is given by:

$$\mathcal{E} = \mathcal{K} + \mathcal{U},$$

The control approach proposed for swinging up the vertical underactuated robots is based on the energy convergence, i.e. the total energy $\mathcal{E}$ follows the desired energy $\mathcal{E}_d$, where $\mathcal{E}_d$ is the energy on the desired unstable equilibrium point, and it is written as:

$$\tilde{\mathcal{E}} = \mathcal{E} - \mathcal{E}_d, \quad (12)$$

which looks like the passivity property (7). Then, we consider the Lyapunov function candidate, as follows:

$$V(t) = \frac{k_E}{2}\tilde{\mathcal{E}}(q,\dot{q})^2 + \frac{k_D}{2}\dot{q}_a^2 + \frac{k_P}{2}\tilde{q}_a^2 + \frac{k_L}{2}\tilde{q}_u^2, \quad (13)$$

where $\tilde{q}_a$ and $\tilde{q}_u$ is the articular error for the final desired position, k_E, k_D, k_P, and k_L are strictly definite positive constants. Since that (13) is a positive semi-definite function, i.e. $V(t) \leq 0$, we verify if (13) has a Lyapunov function. By differentiating (13), and introducing the passivity property (i.e. $\dot{\mathcal{E}} = \dot{q}^T\tau$), we obtain

$$\begin{aligned} \dot{V}(t) &= k_E\tilde{\mathcal{E}}\dot{\mathcal{E}} + k_D\ddot{q}_a\dot{q}_a + k_P\dot{q}_a\tilde{q}_a + k_L\dot{q}_a\tilde{q}_u + k_L\dot{q}_u\tilde{q}_u \\ &= k_E\tilde{\mathcal{E}}\dot{q}_a\tau + k_D\ddot{q}_a\dot{q}_a + k_P\dot{q}_a\tilde{q}_a + k_L\dot{q}_a\tilde{q}_u + k_L\dot{q}_u\tilde{q}_u \\ &= \dot{q}_a\left(k_E\tilde{\mathcal{E}}\tau + k_D\ddot{q}_a + k_P\tilde{q}_a + k_L\tilde{q}_u\right) + k_L\dot{q}_u\tilde{q}_u, \end{aligned} \quad (14)$$

which includes the passivity the property, and stability analysis, we take:

$$\dot{V}(t) = -\dot{q}^T K\dot{q},$$

where K is a semi-definite positive matrix. Let us consider $q = [q_a \;\; q_u]^T$, and propose K as follows:

$$K = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Then the Lyapunov function $\dot{V}$ gives the following equation:

$$\dot{V}(t) = -\left(a\dot{q}_a^2 + 2b\dot{q}_a\dot{q}_u + c\dot{q}_u^2\right). \quad (15)$$

If $b = ac$ then the Lyapunov function can be written as $\dot{V}(t) = -\left(a\dot{q}_a + cq_u\right)^2$. Since (14) is necessarily strictly semi-definite negative, we have induced the Lyapunov function as follows:

$$\begin{gathered}\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \ \leq 0, \\ \dot{V}_1(t) = -a\dot{q}_a^2 - 2b\dot{q}_a\dot{q}_u \leq 0, \\ \dot{V}_2(t) = -b\dot{q}_u^2 \leq 0,\end{gathered} \tag{16}$$

then (14) can be write as follow:

$$\begin{gathered}\dot{q}_a\left(k_E\tilde{\mathcal{E}}\tau + k_D\ddot{q}_a + k_P\tilde{q}_a + k_L\tilde{q}_u\right) \\ = -a\dot{q}_a^2 - 2b\dot{q}_a\dot{q}_u \leq 0, \\ k_L\dot{q}_u\tilde{q}_u = -c\dot{q}_u^2 \leq 0.\end{gathered} \tag{17}$$

By introducing (11) in (17) we obtain:

$$\tau = \frac{-k_D f(q_a, q_u, \dot{q}_a, \dot{q}_u) - (a\dot{q}_a + 2\dot{q}_u + k_p\tilde{q}_a + k_L\tilde{q}_u)\Delta}{k_E\tilde{\mathcal{E}}\Delta + k_D g(q)}. \tag{18}$$

The control law (18) is the solution of the first equation of (17), and we provide the condition $\dot{V}_1(t) \geq \dot{V}_2(t)$, in order to hold the Lyapunov property $\dot{V}(t) \leq 0$. Since (18) is the solution of $\dot{V}_1(t)$, we compensate in τ the condition $\dot{V}_1(t) \geq \dot{V}_2(t)$, and the control law can be written as follows:

$$\begin{aligned}\tau = &\frac{-k_D f(q_a, q_u, \dot{q}_a, \dot{q}_u) - (a\dot{q}_a + 2\dot{q}_u + k_p\tilde{q}_a + k_L\tilde{q}_u)\Delta}{k_E\tilde{\mathcal{E}}\Delta + k_D g(q)} \\ &-c\dot{q}_u - k_L q_u.\end{aligned} \tag{19}$$

This control law proposed acts only to swing up the vertical underactuated 2-DOF robot.

4. STUDY CASE: THE PENDUBOT SYSTEM.

The pendubot as shown in Figure 1, is a benchmark system Fantoni [2002] for underactuated robot, consisting of a double pendulum with an actuator at only the first joint. The pendubot take parameters, the total mass of link 1 is $m_1 = 1.9008m$, the total mass of link 2 is $m_2 = 0.7175m$, the moment of inertia of link 1 is $I_1 = 0.004Kg \cdot m^2$, the moment of inertia of link 2 is $I_1 = 0.005Kg \cdot m^2$, the distance to center of mass of link 1 is $l_{c1} = 0.185m$, the distance to center of mass of link 2 is $l_{c2} = 0.062m$, the length of link 1 is $m_1 = 0.2m$, the length of link 2 is $m_2 = 0.2m$ and the acceleration of gravity constant $g = 9.81m/seg^2$.

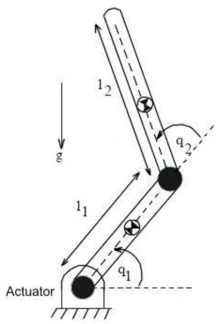

Fig. 1. Pendubot system.

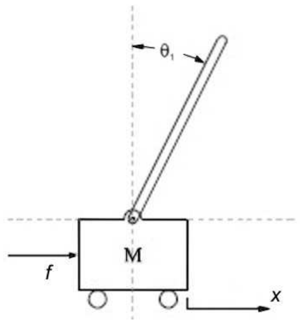

Fig. 2. Cart and pole system.

The model of the motion dynamics is a set of 2 rigid bodies connected and described by a set of generalized coordinates $q \in R^2$. The derivation of the motion equations is given by (1), and by applying the methods of the Lagrange theory, involving explicit expressions of kinetic energy and potential energy we obtain the standard general equation (1). Where the angular positions are involved in q, and angular velocities in $\dot{q}$, and the accelerations is $\ddot{q}$.

From the Euler-Lagrange dynamical model we have that:

$$\begin{gathered}D(q) = \begin{bmatrix} \theta_1 + \theta_2 + \theta_3\cos(q_2) & \theta_2 + \theta_3\cos(q_2) \\ \theta_2 + \theta_3\cos(q_2) & \theta_2 \end{bmatrix}, \\ C(q, \dot{q}) = \theta_3 sin(q_2)\begin{bmatrix} -\dot{q}_2 & -\dot{q}_1 - \dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}, \\ G(q) = \begin{bmatrix} \theta_4 g\cos(q_1) + \theta_5 g\cos(q_1 + q_2) \\ \theta_5 g\cos(q_1 + q_2) \end{bmatrix}. \\ q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \ \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix},\end{gathered} \tag{20}$$

where the following five parameter equations are introduced as follows:

$$\begin{cases} \theta_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\ \theta_2 = m_2 l_{c2}^2 + I_2 \\ \theta_3 = m_2 l_1 l_{c2} \\ \theta_4 = m_1 l_{c1} + m_2 l_1 \\ \theta_5 = m_2 l_{c2} \end{cases} \tag{21}$$

Such introduced a robot is called aa Acrobot if only $\tau_1 \equiv 0$, and is called a Pendubot if only $\tau_2 \equiv 0$.

For the planar two-link with $\tau \equiv 0$ in the Euler-Lagrange system note that (1) has four equilibrium points. The total energy $E(q, \dot{q})$ is different for each of the four equilibrium positions:

Top positions for both links

$$\mathcal{E}((\pi/2), 0, 0, 0) = \mathcal{E}_{Top} = (\theta_4 + \theta_5)g. \tag{22}$$

Low positions for both links

$$\mathcal{E}(-(\pi/2), 0, 0, 0) = \mathcal{E}_{l_1} = -(\theta_4 + \theta_5)g. \tag{23}$$

Mid position: low for link 1 and up for link 2

$$\mathcal{E}(-(\pi/2), \pi, 0, 0) = \mathcal{E}_{mid} = (-\theta_4 + \theta_5)g. \tag{24}$$

Position: up for link 1 and low for link 2.

$$\mathcal{E}((\pi/2), \pi, 0, 0) = \mathcal{E}_{l_2} = (\theta_4 - \theta_5)g. \tag{25}$$

4.1 Stabilizing Control Law

Let us consider the control objective to stabilize the system around its top unstable equilibrium position. In the control following law approach the friction is neglected, because, the tribology forces induce passivity.

The homoclinic orbit. Let us first note in view of (2) and (20), for Euler-Lagrange properties

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}q^T D(q)q + \theta_4 g\sin q_1 + \theta_5 g\sin(q_1 + q_2) \\ &= \frac{1}{2}\theta_2\dot{q}_2^2 + \theta_4 g\sin q_1 + \theta_5 g\sin(q_1 + q_2),\end{aligned} \tag{26}$$

that the following conditions are satisfied

$$\begin{aligned} c_1) &\quad \dot{q}_1 = 0 \\ c_2) &\quad \mathcal{E}(q, \dot{q}) = (\theta_4 + \theta_5)\, g, \end{aligned}$$

then

$$\mathcal{E}(q,\dot{q}) = \frac{1}{2}\theta_2 q_2^2 + \theta_4 g \sin q_1 + \theta_5 g \sin(q_1+q_2) \\ = \theta_4 g + \theta_5 g. \tag{27}$$

From the above, it follows that if $q_1 \neq \pi/2$ then $\dot{q}_2^2 > 0$. In addition to conditions c_1) and c_2) we also have the condition $c3$) $q_1 + q_2 \approx \frac{\pi}{2}$, then (27) gives

$$\frac{1}{2}\theta_2 \dot{q}_2^2 = \theta_4(\cos q_2 + g). \tag{28}$$

The above equation defines a very particular trajectory that corresponds to a homoclinic [1] orbit. This means that the link 2 angular position moves clockwise or counter-clockwise until it reaches the equilibrium position $(q_2, \dot{q}_2) = (0, 0)$ which can arrive at the infinity. Thus, our objective can be reached if the system be brought to the orbit (28) for $\dot{q}_1 = 0$ and $q_1 \approx \pi/2$ at the same time that $q_1 - q_2 \approx \pi/2$. Bringing the system to this homoclinic orbit, it solves the "swing up" problem. By guaranteing convergence to the above homoclinic orbit, we guarantee that the trajectory will enter the basing of attraction of any balancing controller.

Stabilizing around the homoclinic orbit. The passivity property of the system suggest the use the total energy $\mathcal{E}$ in (26) in the controller design. Let us consider $\tilde{q}_1 = (q_1 - \pi/2)$, $\tilde{q}_2 = q_1 + q_2 - \pi/2$ in other words $\tilde{q}_2 = \tilde{q}_1 - q_2$ and $\tilde{\mathcal{E}} = \mathcal{E} - \mathcal{E}_{Top}$, we wish bring to zero $\tilde{q}_1$, $\tilde{q}_2$ and $\tilde{\mathcal{E}}$. We propose the following Lyapunov function candidate (13), where $q_a = q_1$, $q_u = q_2$, k_E, k_D, k_P and k_L are strictly positive constants to be defined later. Note that $V(t)$ is a positive semi-definite function. Where $\tau = [\tau_1, 0]^T$ in the pendubot system. Then,

$$\dot{\mathcal{E}} = \dot{q}^T \tau = \dot{q}_1 \tau_1. \tag{29}$$

By differentiating $V(t)$ and using (14), we obtain

$$\dot{V}(t) = \dot{q}_1 \left(k_E \tilde{\mathcal{E}} \tau_1 + k_D \ddot{q}_1 + k_P \tilde{q}_1 + k_L \tilde{q}_2 \right) + k_L \dot{q}_2 \tilde{q}_2, \tag{30}$$

which means that the system is passive, by taking τ as the input and q_1 as the output.

Let us now compute $\ddot{q}_1$ from (10). The inverse of $D(q)$ can be obtained from (20) is given by

$$D^{-1}(q) = \frac{1}{\Delta} \begin{bmatrix} \theta_2 & -\theta_2 - \theta_3 \cos q_2 \\ -\theta_2 - \theta_3 \cos q_2 & \theta_1 + \theta_2 + 2\theta_3 \cos q_2 \end{bmatrix}, \tag{31}$$

where $\Delta = det(D(q)) = \theta_1\theta_2 - \theta_2^3 cos^2 q_2$, and $\ddot{q}_1$ can thus be written as

$$\ddot{q}_1 = \frac{1}{\Delta} \left[\theta_2\theta_3 \sin q_2 (\dot{q}_1 + \dot{q}_2)^2 + \theta_3^2 \cos q_2 \sin q_2 \dot{q}_1^2 \right. \\ \left. -\theta_2\theta_4 g \cos q_1 + \theta_3\theta_5 g \cos q_2 \cos(q_1+q_2) + \theta_2\tau_1 \right], \tag{32}$$

To reduce the expressions, we will consider

$$f(q,\dot{q}) = \left[\theta_2\theta_3 \sin q_2 (\dot{q}_1 + \dot{q}_2)^2 + \theta_3^2 \cos q_2 \sin q_2 \dot{q}_1^2 \right. \\ \left. -\theta_2\theta_4 g \cos q_1 + \theta_3\theta_5 g \cos q_2 \cos(q_1+q_2) + \theta_2\tau_1 \right],$$

and thus

$$\ddot{q}_1 = \frac{1}{\Delta} \left[\theta_2 \tau_1 + f(q,\dot{q}) \right]. \tag{33}$$

By introducing the former equation in (30) one has

$$\dot{V}(t) = \dot{q}_2 \tilde{q}_2 \\ + \dot{q}_1 \left\{ \tau_1 \left(k_E \tilde{\mathcal{E}} + \frac{K_D \theta_2}{\Delta} \right) + \frac{K_D f(q,\dot{q})}{\Delta} + k_p \tilde{q}_1 + k_L \tilde{q}_2 \right\}, \tag{34}$$

which will lead to (15). Then, control law (18) can be written as:

$$\tau = \frac{-k_D f(q_a, a_u, \dot{q}_a, \dot{q}_u) - (a\dot{q}_a + 2\dot{q}_u + k_p \tilde{q}_a + k_L \tilde{q}_u)\Delta}{k_E \tilde{\mathcal{E}} \Delta + k_D g(q).} \tag{35}$$

Finally, with the condition $\dot{V}_1(t) \geq \dot{V}_2(t)$. We chose the K matrix gains as:

$$K = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

then the control law can be written as:

$$\tau = \frac{-k_D f(q,\dot{q}) - (a\dot{q}_1 + 2b\dot{q}_2 + k_p \tilde{q}_1 + k_L \tilde{q}_2)\Delta}{k_E \tilde{\mathcal{E}} \Delta + k_D \theta_2} \\ -c\dot{q}_2 - K_L \tilde{q}_2 \tag{36}$$

We chose the gains $k_P = 0.72$, $k_D = 0.93$, $k_E = 0.65$, $k_L = 1.01$, found by trial-error.

4.2 Stabilization around an unstable equilibrium point.

Using the linearized system, in the upper unstable equilibrium point i.e. $[q_1, q_2, \dot{q}_1, \dot{q}_2] = [\pi/2, 0, 0, 0]$. The system (2) can be stabilized via a linear control, since that $\det(B|AB|A^2B|A^3B) = \frac{g^2\theta_5^2\theta_3^2}{(\theta_1\theta_2 - \theta_3^2)^4}$, and then the linearized system is controllable. We chose the Q and R matrix gains as:

$$R = 2.5 \\ Q = \begin{bmatrix} 1000 & -500 & 0 & 0 \\ -500 & 1000 & 0 & 0 \\ 0 & 0 & 1000 & -500 \\ 0 & 0 & -500 & 1000 \end{bmatrix} \tag{37}$$

we obtain the control law:

$$\tau = -58.8019\tilde{q}_1 - 58.2007 q_2 - 12.0533\dot{q}_1 \\ -7.9339\dot{q}_2, \tag{38}$$

and the balance switch function is given by

$$\left\{ \left| q_1 - \frac{\pi}{2} \right| < \delta_1 \right\} \&\& \left\{ \left| \frac{\pi}{2} - (q_1+q_2) \right| < \delta_2 \right\}, \tag{39}$$

were $\delta_1 = 0.2rad$ and $\delta_2 = 0.3rad$.

5. STUDY CASE: CART AND POLE SYSTEM.

The cart on pole as shown in Figure 2, has been studied as a typical example of underactuated mechanical systems see e.g. Ordaz [2007]-Spong [1994]. We considered the cart and pole parameters as: the total mass of link 1 $m_1 = 1.9008$ m, the total mass of link 2 $m_2 = 0.7175$ m, the moment of inertia of link 1 $I_1 = 0.004$ $Kg \cdot m^2$, the moment of inertia of link 2 $I_1 = 0.005$ $Kg \cdot m^2$, the

[1] **A homoclinic orbit** is a trajectory of a dynamical system that tends to the same invariant set (equilibrium, fixed point, periodic orbit etc.) as time $t \to \pm\infty$

distance to center of mass of link 1 $l_{c1} = 0.185$ m, the distance to center of mass of link 2 $l_{c2} = 0.062$ m, the length of link 1 $m_1 = 0.2$ m, the length of link 2 $m_2 = 0.2$ m and the acceleration of gravity $g = 9.81$ m/seg^2. The dynamic model is:

$$q = \begin{bmatrix} \theta \\ x \end{bmatrix}, \ \tau = \begin{bmatrix} \tau \\ 0 \end{bmatrix}, D(q) = \begin{bmatrix} M+m & ml\cos(\theta) \\ ml\cos(\theta) & ml^2 \end{bmatrix}$$
$$C(q,\dot{q}) = \theta_3 sin(q_2) \begin{bmatrix} 0 & ml\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix} \quad (40)$$
$$g(q) = \begin{bmatrix} 0 \\ -mgl\sin(\theta) \end{bmatrix}$$

For the cart on pole system with $\tau \equiv 0$ in the Euler-Lagrange system note that (1) has two equilibrium points. $(\theta, x, \dot{\theta}, \dot{x}) = (\pi, *, 0, 0)$ the stable equilibrium point and $(\theta, x, \dot{\theta}, \dot{x}) = (0, *, 0, 0)$ is the stable equilibrium position that also we want to avoid. The control objective is to stabilize the system around its top unstable equilibrium position.

5.1 Stabilizing Control Law

Let us consider the control objective to stabilize the system around its top unstable equilibrium position. In the control following law approach the friction is neglected, because, the tribology forces induce passivity.

The homoclinic orbit Let us first note in view of (20), for Euler-Lagrange properties

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} q^T D(q) q + mgl(\cos(\theta)) \\ &= \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta} + \frac{1}{2}(I+ml^2)\dot{\theta}^2 \\ &+ mgl(\cos(\theta)) \end{aligned} \quad (41)$$

and the following conditions are satisfied

$$\begin{aligned} c_1) & \quad \dot{\theta} = 0 \\ c_2) & \quad \mathcal{E}(q,\dot{q}) = mgl \end{aligned}$$

From the above, it follows that if $\theta \neq 0$ then $\dot{x}^2 > 0$. In addition to conditions c_1) and c_2) we also have condition $c3$) $\theta + x \approx 0$, then c_2) gives

$$\frac{1}{2} ml\dot{\theta}^2 = mgl(\cos(\theta) - 1) \quad (42)$$

The above equation defines a very particular trajectory that corresponds to a homoclinic orbit.

5.2 Stabilizing around the homoclinic orbit

The passivity property of the system suggest as the use of the total energy $\mathcal{E}$ in the controller design. Let us consider $\tilde{\mathcal{E}} = \mathcal{E} - \mathcal{E}_{Top}$, we wish bring to zero $\tilde{q}_1$, $\tilde{q}_2$ and $\tilde{\mathcal{E}}$. We propose the following Lyapunov function candidate

$$V(q,\dot{q}) = \frac{k_E}{2}\tilde{\mathcal{E}}(q,\dot{q})^2 + \frac{k_D}{2}\dot{x}^2 + \frac{k_P}{2}x^2 + \frac{k_L}{2}\theta^2 \quad (43)$$

and following the pendubot proceeds, we obtain the control law:

$$\tau_1 = \frac{-k_D F(q,\dot{q}) - (x - 2\dot{\theta} + k_p\tilde{x} + k_L\theta)\Delta}{k_E\tilde{\mathcal{E}}\Delta + k_D} - \dot{\theta} - k_L\theta \quad (44)$$

5.3 Stabilization around an unstable equilibrium point.

For the system stabilization we design an optimal linear quadratic controller. We chose the Q and R matrix gains as the pendubot gains (37), and the balance switch function is given by (36), where $\delta_1 = 0.35$ rad and $\delta_2 = 0.25$ rad. We chose the gains $k_P = 41$, $k_D = 80$, $k_E = 15.51$, $k_L = 15.012$, found by trial-error.

6. SIMULATION RESULTS

In order to illustrate the performance of the proposed control law based on the system energy, we have performed some numerical simulations on MATLAB.

6.1 First energy shaping (Pendubot).

The figures 3, 4 illustrate the swinging up and stabilization around the critical unstable equilibrium point, we start the simulation on initial conditions $q_1 = -\frac{\pi}{2}$, $\dot{q}_1 = 0$, $q_2 = 0$, $\dot{q}_2 = 0$, the stabilization becomes around that two seconds, and the stability is shows in the figure 7.

6.2 Second energy shaping (Cart and pole).

The figures 5, 6 illustrate the swinging up and stabilization around the critical unstable equilibrium point, we start the simulation for initial conditions $x = 0.5$, $\dot{x} = 0$, $\theta = \pi$, $\dot{\theta} = 0$, the stabilization becomes around that five seconds, and the stability is shows in the figure 7.

The simulation results show that the control proposed law brings the state of system to the homoclinic orbit (see figure 4 and 6). Note that $\tilde{\mathcal{E}}$ goes to zero, i.e. the energy $\mathcal{E}$ goes to energy at the upright position ($\mathcal{E}_{Top}$). The Lyapunov function V is always decreasing and converges to zero (see figure 7).

7. CONCLUDING REMARKS

In this paper, a complete approach in order to swinging up a and balancing a 2-DOF vertical underactuated robot has been introduced. Our approach is based on the system energy analysis and passivity properties.

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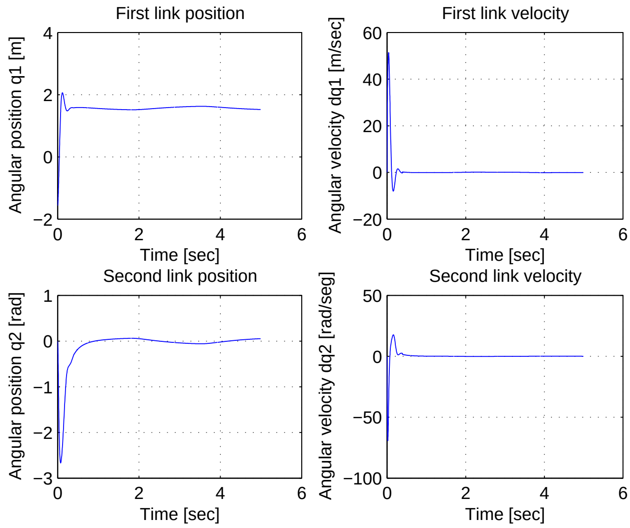

Fig. 3. Pendubot position and velocity.

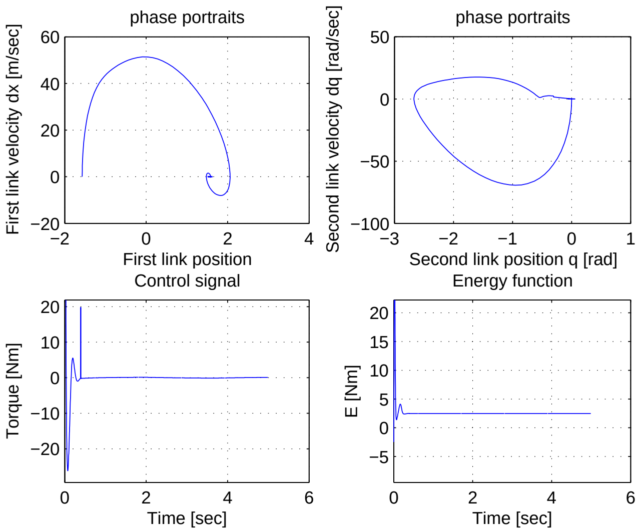

Fig. 4. Control signal, energy, and phase portraits (pendubot).

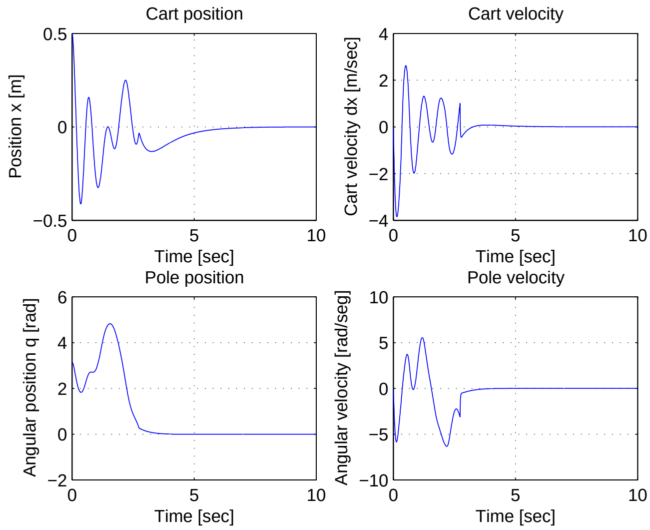

Fig. 5. Cart on pole position and velocity.

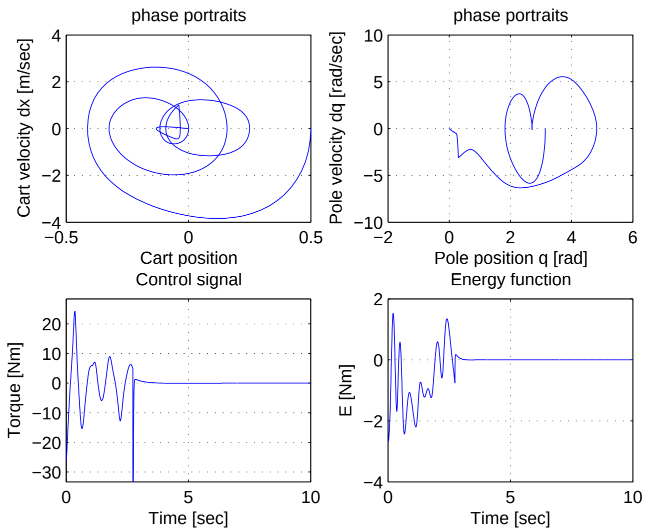

Fig. 6. Control signal, energy, and phase portraits (cart on pole)

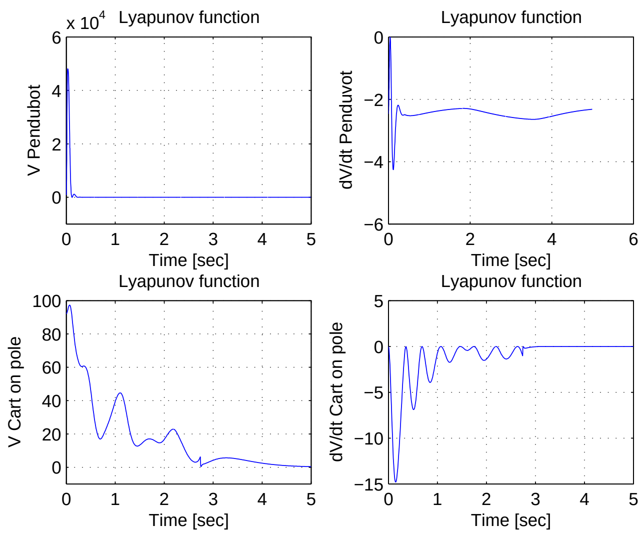

Fig. 7. Lyapunov functions (pendubot and cart on pole).