# Operator Based Robust Nonlinear Control System Design of MIMO Nonlinear Feedback Control Systems

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**Abstract:** In this paper, operator based robust nonlinear control system design for multi-input multi-output nonlinear feedback control systems is proposed, that is, robust stability of the MIMO systems is studied by using operator based robust right coprime factorization approach. Some sufficient conditions for the MIMO nonlinear systems to be robust stable are derived. As a result, robust nonlinear control is designed for the MIMO systems. An example is given to demonstrate the theoretical analysis.

## 1. INTRODUCTION

Coprime factorization approach has been considered as an effective tool for studying operator based robust nonlinear control system design of nonlinear feedback control systems, including robust stabilization analysis, output tracking design, and fault detection and so on. Moreover, right coprime factorization approach provides a convenient framework for the nonlinear control system design, which has attracted much attention due to its effectiveness (see Chen and Han, 1998; Deng and Inoue, 2008; Deng et al., 2007; Deng et al., 2006; Deng et al., 2004; Deng, Inoue and Yanou, 2007). A brief summary on the relevant studies is presented below to lay a foundation for the paper.

Coprime factorization concept has been introduced into nonlinear feedback control systems since 1980s (see Banos, 1994; Hammer, 1984; Vidyasagar, 1985). Robust right coprime factorization of nonlinear plants under perturbation was studied in Chen and Han (1998). Output tracking problem with perturbed nonlinear plants (see Deng et al., 2004) has been considered by extending the condition given in Chen and Han (1998), and perturbed Bezout identity was given for the above class of nonlinear plants. Output tracking problem with different spaces of reference input and output was discussed (see Deng et al., 2006). Recently, by using robust right coprime factorization approach, fault detection in a thermal process control system with input constraints was solved (see Deng et al., 2007). Stable robust feedback control system design for unstable plants with input constraints was studied in Deng, Inoue and Yanou (2007), and networked based nonlinear control for an aluminum plate thermal process with time-delays was designed in Deng and Inoue (2008). However, the robust nonlinear control system design for multi-input multi-output nonlinear control systems have seldom been studied due to the difficulty in dealing with coupling effects.

Multi-input multi-output nonlinear feedback control system design is studied in this paper. Robust stability is considered by using operator based robust right coprime factorization. The remainder of the paper is organized as follows. In Section 2, some knowledge about operator theory is introduced, and one class of MIMO nonlinear feedback systems is designed. In Section 3, operator based robust nonlinear system design is considered and some sufficient conditions for the MIMO systems to be stable are derived. A numerical example is presented to demonstrate the proposed design schemes in Section 4 and conclusion is drawn in Section 5.

## 2. PRELIMINARIES AND PROBLEM STATEMENT

### *2.1 Preliminaries*

In this section, some knowledge about operator theory for feedback control systems is introduced (see Chen and Han, 1998; de Figueiredo and Chen, 1993; Deng et al., 2006).

Let $X$ and $Y$ be linear spaces over the field of real numbers, $X_s$ and $Y_s$ are normed linear subspaces, called the *stable spaces* of $X$ and $Y$. Let $Q : X \to Y$ be an operator, the *domain* and *range* of $Q$ is denoted by $\mathcal{D}(Q)$ and $\mathcal{R}(Q)$, respectively. We always assume that $\mathcal{D}(Q) = X$ with $\mathcal{R}(Q) \subset Y$. In this note, an operator is said to be bounded input bounded output (BIBO) stable or simply, stable if $Q(X_s) \subset Y_s$.

**Definition 1**. Let $S(X, Y)$ be the set of stable operators mapping from $X$ to $Y$. Then, $S(X, Y)$ contains a subset defined by

$$\begin{aligned} \mathcal{U}(X, Y) = \{&M : M \in S(X, Y), \\ &M \text{ is invertible with } M^{-1} \in S(Y, X)\} \end{aligned} \qquad (1)$$

Elements of $\mathcal{U}(X, Y)$ are called *unimodular* operators.

Consider a normal nonlinear feedback control system shown in Fig.1, $U$ and $V$ is used to denote the input and output spaces of a given plant operator $P$, i.e., $P : U \to V$.

**Definition 2**. The given plant operator $P : U \to V$ is said to have a *right factorization*, if there exist a linear space

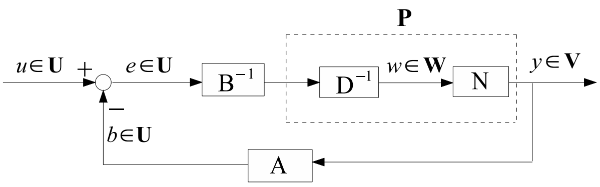


Fig. 1. A nonlinear feedback system

$W$ and two stable operators $D : W \to U$ and $N : W \to V$ such that $D$ is invertible from $U$ to $W$, and $P = ND^{-1}$ on $U$. Such a factorization of $P$ is denoted by $(N, D)$ and space $W$ is called a $quasi-state\ space$ of $P$ (The $left\ factorization$ means $P = D^{-1}$).

**Definition 3**. Let $(N, D)$ be a right factorization of $P$. The factorization is said to be *coprime*, or $P$ is said to have a $right\ coprime\ factorization$, if there exist two stable operators $A : V \to U$ and $B : U \to U$, satisfying the *Bezout identity*

$$AN + BD = M, \text{ for some } M \in \mathcal{U}(W, U), \tag{2}$$

where $B$ is invertible. Usually, $P$ is unstable and $(N, D, A, B)$ are to be determined (the so-called system design problem).

**Definition 4**. The feedback control system shown in Fig.1 is said to be $well-posed$, if for every input signal $u \in U$, all the signals in the system (i.e., $e, z, b$ and $y$) are uniquely determined.

**Definition 5**. The feedback control system shown in Fig.1 is said to be *overall stable*, if $u \in U_s$, implies that $y \in V_s$, $z \in W_s$, $e \in U_s$ and $b \in U_s$.

**Definition 6**. Let $U^e$ and $V^e$ be two extended linear spaces, which are associated respectively with two given Banach space $U_B$ and $V_B$ of measurable functions defined on the time domain $[0, \infty)$, where a Banach space is complete vector with a norm. Let $D^e$ be a subset of $U^e$. A nonlinear operator $A : D^e \to V^e$ is called a *generalized Lypschitz operator* on $D^e$ if there exists constant $L$ such that:

$$\|[A(x)]_T - [A(\tilde{x})]_T\|_{V_B} \leq L\|x_T - \tilde{x}_T\|_{U_B}. \tag{3}$$

*2.2 Problem statement*

To consider nonlinear control system design of MIMO nonlinear feedback control systems by using operator based robust right coprime factorization, one class of MIMO nonlinear control systems is described and problem setup is given in this subsection.

Consider the multi-input multi-output nonlinear closed-loop systems shown in Fig.2. Where, $\mathbf{P}$ are given nominal nonlinear plants of the MIMO systems. Let input space, output space, quasi-state space be $U$, $V$, $W$, and plants $\mathbf{P} : U \to V$. Let the signals of input, error, control input, quasi-state and plant output be $\mathbf{u} = (u_1, u_2, \cdots, u_n) \in U$, $\mathbf{e} = (e_1, e_2, \cdots, e_n) \in U$, $\mathbf{z} = (z_1, z_2, \cdots, z_n) \in U$, $\mathbf{w} = (w_1, w_2, \cdots, w_n) \in W$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in V$, respectively.

For MIMO systems, the given nonlinear plants $\mathbf{P} : U \to V$ have $right\ factorization$ means $\mathbf{P} = \mathbf{N}\mathbf{D}^{-1}$ and $\mathbf{P}$ have $right\ coprime\ factorization$ indicates that the Bezout identity

$$\mathbf{AN} + \mathbf{BD} = \mathbf{M}, \text{ for some } \mathbf{M} \in \mathcal{U}(W, U) \tag{4}$$

is satisfied, where, $\mathbf{A} : V \to U$, $\mathbf{B} : U \to U$, $\mathbf{D} : W \to U$ and $\mathbf{N} : W \to V$ are stable operators, $\mathbf{B}, \mathbf{D}$ are invertible. Usually, the given plants $\mathbf{P}$ are unstable.

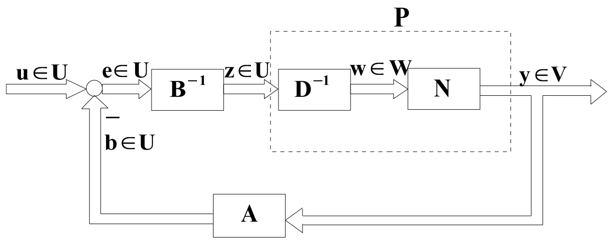


Fig. 2. The MIMO nonlinear feedback systems

**Remark.** Assume that every input signal $u_i \in \mathbf{u}$ belong to different subspace $U_i$ of input space $U$, namely, $u_i \in U_i \subset U$. Similarly, error signal $e_i \in U_i \subset U$, control input signal $z_i \in U_i \subset U$, quasi-state signal $w_i \in W_i \subset W$, and plant output signal $y_i \in V_i \subset V$. For brevity, we mainly consider two-input two-output nonlinear plants.

For two-input two-output feedback systems, when the systems are constructed by two independent subsystems, that is, there is no coupling effects between the two plants $P_i$, then the plants outputs $y_i$ $(i = 1, 2)$ only have relations with the corresponding signals $u_i, e_i, z_i, w_i$ $(i = 1, 2)$. Assume that the nominal plant $P_i$ $(i = 1, 2)$ have right coprime factorization and satisfy the Bezout identity. In this note, every operator can be divided as the following forms, $\mathbf{A} = (A_1, A_2)$, $\mathbf{B} = (B_1, B_2)$, $\mathbf{D} = (D_1, D_2)$, $\mathbf{N} = (N_1, N_2)$, and the Bezout identities are,

$$\begin{aligned} &A_iN_i + B_iD_i = M_i, \\ &\text{for some } M_i \in \mathcal{U}(W, U), \quad i = 1, 2. \end{aligned} \tag{5}$$

Where, $\mathcal{U}(W, U)$ is the set of unimodular operators, $A_i : V_i \to U_i$, $B_i : U_i \to U_i$, $D_i : W_i \to U_i$ and $N_i : W_i \to V_i$ are stable, $B_i$ and $D_i$ are invertible. $A_i$, $B_i^{-1}$ $(i = 1, 2)$ represent the feedback and feedforward controllers of the subsystems, respectively. $M_i$ of the Bezout identity is equal to the operator $M_i : W_i \to U_i$ of the overall subsystem. If $M_i$ is unimodular operator, thus the operator $M_i^{-1} : U_i \to W_i$ of the overall subsystem is stable, $N_i$ is stable because of right coprime factorization. Finally the systems are internally stable because the two subsystems are stable.

Generally speaking, there exist coupling effects between the two subsystems. In order to demonstrate the coupling effects, suppose that the Bezout identity as (5) is satisfied for every plant $P_i$, and exist two internal operators $G_i$ $(i = 1, 2)$ concerning with the coupling effects. The detail description is presented below.

Assume that output signal of operator $B_i^{-1}$ is $x_i \in U_i$, namely, $x_i(t) = B_i^{-1}e_i(t)$, and there exist two operators $G_1 : U_2 \to U_1$ and $G_2 : U_1 \to U_2$ satisfying that control input signals

$$\begin{aligned} z_1(t) &= x_1(t) + G_1(x_2)(t), \\ z_2(t) &= x_2(t) + G_2(x_1)(t), \end{aligned}$$

such that the coupling effects are concerned with internal operator $G_i$. The quasi-state signal is $w_i(t) = D_i^{-1}(z_i)(t)$, plant output signal is $y_i(t) = N_i(w_i)(t)$, thus the systems described here are multi-input multi-output nonlinear systems.

The main objective of this paper is to design an operator based robust nonlinear control system design for the MIMO nonlinear systems described above.

## 3. ROBUST NONLINEAR CONTROL SYSTEM DESIGN

In this section, we study the MIMO nonlinear control system design by using robust right coprime factorization approach.

For the MIMO nonlinear feedback systems shown in Fig.2, based on the description in Section 2, the nominal plant $P_i$ of every subsystem has right coprime factorization as (5), and coupling effects between the two subsystems are concerning with the internal operators $G_i$ $(i = 1, 2)$.

One method for studying the MIMO nonlinear systems is consider the coupling effects as perturbation effects from the uncertainties of $D_i$, then, the MIMO nonlinear systems are divided into two independent perturbed subsystems. As studied before (see Deng et al., 2006; Deng and Inoue, 2008), the similar conditions as given in Theorem 1 of Deng et al. (2006) can ensure the stability of the systems. However, it is difficult to design the systems due to the uncertainties. Therefore, in order to design the MIMO systems more detailed, the internal operators are analyzed as follows.

**Theorem 1**. For the MIMO nonlinear feedback control systems described above, assume the systems are well-posed, and the nominal plants $P_i$ $(i = 1, 2)$ have right coprime factorization as (5). Suppose that $G_i$ $(i = 1, 2)$ are stabilizable, that is, $G_1$ and $G_2$ can be stabilized by $B_2^{-1}$ and $B_1^{-1}$, respectively. If $M_i^{-1} : U_i \to W_i$, $A_i N_i : W_i \to U_i$ $(i = 1, 2)$, $B_1 G_1 B_2^{-1} : U_2 \to U_1$ and $B_2 G_2 B_1^{-1} : U_1 \to U_2$ are generalized Lypschitz operators have inequalities as (3) with Lypschitz constants $L_1$, $L_2$, $L_3$ and $L_3$, where $L_1 L_2 L_3 < 1$. Then the systems are overall stable.

**proof.** Since the Bezout identity satisfied, the systems have the following relations

$$u_1(t) = M_1(w_1)(t) + B_1 G_1 B_2^{-1}(u_2 - A_2 N_2 w_2)(t),$$
$$u_2(t) = M_2(w_2)(t) + B_2 G_2 B_1^{-1}(u_1 - A_1 N_1 w_1)(t).$$

Based on Contraction Mapping Theorem (see de Figueiredo and Chen, 1993), it can be derived that for any $(u_1, u_2) \in (U_{1s}, U_{2s}) \subset U_s$, the signal pair $(w_1, w_2)$ is uniquely determined by $(u_1, u_2)$, and $(w_1, w_2) \in (W_{1s}, W_{2s}) \subset W_s$. Moreover, it follows that $y_i \in V_{is} \subset V_s$, $e_i \in U_{is} \subset U_s$, $z_i \in U_{is} \subset U_s$. Then, the systems are overall stable. □

If $G_i$ $(i = 1, 2)$ can not be stabilized, then the stability of MIMO systems is difficult to ensure because of unstable internal operators. For this reason, right or left factorization is an effective way to solve this problem.

Assume that operators $G_i$ can be divided into two parts, just as $G_i = T_i S_i^{-1}$. Here, $T_1 : W_2 \to U_1$, $S_1 = D_2 : W_2 \to U_2$, $T_2 : W_1 \to U_2$ and $S_2 = D_1 : W_1 \to U_1$ are stable. That is, $G_i$ has right factorization. Assume that $I - G_1 G_2$, $I - G_2 G_1$ are invertible, and $B_1, B_2$ can stabilize $(I - G_1 G_2)^{-1}$, $(I - G_2 G_1)^{-1}$ in the sense that $B_1^* = B_1(I - G_1 G_2)^{-1}$ and $B_2^* = B_2(I - G_2 G_1)^{-1}$ are stable. If operators $A_i, B_i^*$ satisfy the Bezout identity, then the MIMO nonlinear systems are stable provided that the following conditions are satisfied.

**Theorem 2**. Consider the MIMO nonlinear feedback control systems shown in Fig.2. Assume the systems are well-posed, and the nominal plants $P_i$ $(i = 1, 2)$ have right coprime factorization as (5). When operators $G_i$ $(i = 1, 2)$ have right factorization as $G_i = T_i S_i^{-1}$, here, $S_1 = D_2$, $S_2 = D_1$ and $T_i$ are stable. Suppose $B_i^*$ is stable, $A_i, B_i^*$ satisfy the Bezout identity

$$A_i N_i + B_i^* D_i = \tilde{M}_i, \tilde{M}_i \in \mathcal{U}(W, U), i = 1, 2. \qquad (6)$$

If $\tilde{M_i}^{-1} : U_i \to W_i$ $(i = 1, 2)$, $B_1^* T_1 : W_2 \to U_1$, $B_2^* T_2 : W_1 \to U_2$ are generalized Lypschitz operators have inequalities as (3) with the Lypschitz constants $L_1$, $L_2$ and $L_2$, where $L_1 L_2 < 1$. Then the systems are overall stable.

**Proof.** According to the assumption, the internal operators are factorized as $G_1 = T_1 D_2^{-1}$ and $G_2 = T_2 D_1^{-1}$, then the signals of the feedback systems have the following relations

$$D_1(w_1)(t) - x_1(t) = T_1(w_2)(t) - G_1 G_2(x_1)(t),$$
$$D_2(w_2)(t) - x_2(t) = T_2(w_1)(t) - G_2 G_1(x_1)(t).$$

If the Bezout identity condition (5) is satisfied, then

$$u_1(t) = \tilde{M}_1(w_1)(t) - B_1^* T_1(w_2)(t),$$
$$u_2(t) = \tilde{M}_2(w_2)(t) - B_2^* T_2(w_1)(t). \qquad (7)$$

Since $B_i^*$ $T_i$ are stable, $\tilde{M}_i$ is unimodular operator and the Lypschitz condition is satisfied, then for any $u_i \in U_s$, it has that $w_i \in W_s$. Then, the systems are overall stable. □

The internal operator $G_i$ has right factorization, it follows that the coupling effects between the two subsystems are equal in a certain degree that one quasi-state signal $w_i$ of plant $P_i$ affecting on the control input signal $z_j (i \neq j, i, j = 1, 2)$ of another subsystem. By this means, the coupling effects are transformed and can be controlled stably. This ensures the robust stability of the MIMO systems.

In the same way, if the operators $G_i$ ($G_1 : U_2 \to U_1$, $G_2 : U_1 \to U_2$) can be factorized as $G_i = T_i^{-1} S_i$ ($T_i = D_i : W_i \to U_i$, $S_1 : U_2 \to W_1$, $S_2 : U_1 \to W_2$, $T_i$, $S_i$ $(i = 1, 2)$ are stable), then the coupling effects between two plants are changed that the control input signal of one plant affecting the quasi-state of the other plant, the stability of the feedback systems still can be proved.

As for multi-input multi-output nonlinear systems, the coupling effects to one plant are the sum of the coupling effects from other plants, and the amount of relations for the internal signals is the same as the amount of the inputs. As a result, Theorems 1 and 2 can be extended to n-input n-output nonlinear systems.

## 4. NUMERICAL EXAMPLE

The purpose of this section is to demonstrate the benefit of the proposed design scheme, where we choose a two-input two-output nonlinear systems as follows. The first plant is given in Chen and Han (1998).

$$P_1(\tilde{u}_1)(t) = \int_0^t \tilde{u}_1^{1/3}(\tau)d\tau + e^{t/3}\tilde{u}_1^{1/3}(t),$$
$$N_1(w_1)(t) = \int_0^t e^{-\tau/3}w_1^{1/3}(\tau)d\tau + w_1^{1/3}(t),$$
$$D_1(w_1)(t) = e^{-t}w_1(t),$$

the other plant is given as follows,

$$P_2(\tilde{u}_2)(t) = \int_0^t \tilde{u}_2(\tau)d\tau,$$
$$N_2(w_2)(t) = \int_0^t (1+e^{\tau})^{-1}w_2(\tau)d\tau,$$
$$D_2(w_2)(t) = (1+e^t)^{-1}w_2(t).$$

And the internal operators between the two plants are

$$G_1(\tilde{u})(t) = I(\tilde{u})(t),$$
$$G_2(\tilde{u})(t) = (1+e^{-t})^{-1}(\tilde{u})(t).$$

Where, $\tilde{u}_i \in U_i$ with $P_i(\tilde{u}_i) \in V_i$, $U_1 = U_2 = U = C_{[0,\infty)}$, $V_1 = \{u + e^{t/3}u' | u \in C^1_{[0,\infty)}\} \subset U$, $V_2 = U$, $V = V_1 \cup V_2$, it follows that $D(P_i) = U$ and $R(P_i) \subset V$. First, it must be pointed out that $P_1$, $P_2$ are unstable. Then, it can be proved that $P_i$ has a right factorization $P_i = N_i D_i^{-1}$ with $N_i : W \to V$ and $D_i : W \to U$, and $N_i, D_i$ are stable, where we choose $W = U$, $W_s = U_s$.

The controllers are designed as

$$A_1(y)(t) = \begin{cases} (e^t - 1)(u')^3(t), \text{if } y = u + e^{t/3}u' \\ 0, \quad \text{otherwise}, \end{cases}$$
$$B_1(u)(t) = I(u)(t),$$
$$A_2(y)(t) = e^t y'(t),$$
$$B_2(u)(t) = I(u)(t).$$

It can be verified that the above operators are stable and they satisfy the Bezout identity,

$$A_i N_i(w_i)(t) + B_i D_i(w_i)(t) = I(w_i)(t),$$
$$\text{for all } w_i \in W, i = 1, 2.$$

Since the internal operators are stable, and the conditions of Theorem 1 are satisfied, then the two-input two-output systems are designed stable. In the same way, Theorem 2 can be used in this example. Based on the proposed design scheme, operator $G_i$ is divided into two parts

$$S_1(\tilde{w}_2)(t) = (1+e^t)^{-1}\tilde{w}_2(t),$$
$$S_2(\tilde{w}_1)(t) = e^{-t}\tilde{w}_1(t),$$
$$T_1(\tilde{w}_2)(t) = (1+e^t)^{-1}\tilde{w}_2(t),$$
$$T_2(\tilde{w}_1)(t) = (1+e^t)^{-1}\tilde{w}_1(t).$$

By Theorem 2, it can be derived that

$$\tilde{M}_1(w_1)(t) = 2I(w_1)(t),$$
$$\tilde{M}_2(w_2)(t) = (2I - (1+e^t)^{-1})(w_2)(t).$$

$\tilde{M}_i (i = 1, 2)$ are unimodular operators and the proposed conditions of Theorem 2 are satisfied, then the systems are overall stable. This example demonstrates the effectiveness of Theorems 1 and 2.

## 5. CONCLUSION

One class of multi-input multi-output nonlinear control system design based on operator theory was studied in this paper. The system stabilization has been discussed by using robust right coprime factorization approach, and some sufficient conditions for the system to be robustly stable were derived. The effectiveness of the proposed design schemes were also confirmed by a numerical example.

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