COMBINED ATTITUDE CONTROL OF AN UNDERACTUATED HELICOPTER EXPERIMENTAL SYSTEM

Mingcong Deng *, **Akira Inoue** *,
and Tatsunori Shimizu *

* *Dept. Systems Eng., Okayama University*
3-1-1 Tsushima-Naka, Okayama, Japan 700-8530
{*deng, inoue*}*@suri.sys.okayama-u.ac.jp*

Abstract: In this paper, combined attitude control of an underactuated helicopter experimental system is considered. The controlled helicopter experimental system has two inputs and three outputs, namely, this system is underactuated. The combined attitude controller includes a nonlinear MIMO controller based on adaptive sliding mode control and non-adaptive nonlinear controllers. Control system stability is guaranteed by Lyapunov function based proof. Comparing simulation between the existed design method and the proposed design method shows the effectiveness of the proposed method.

Keywords: Helicopter, underactuated system, modeling, control

1. INTRODUCTION

Controlling an underactuated helicopter has received a lot of attention, due to the fact that the helicopter system is an underactuated nonlinear system. That is, the number of control inputs is less than the number of the nonlinear system outputs. This paper develops a combined attitude control for an underactuated helicopter experimental system, where the controlled system is a three-degree of freedom helicopter system that has two inputs and three outputs. So far, concerning with this topic of underactuated system control, many works have been developed (for example, [1] - [4], etc.) for different controlled systems. Especially, PD-based control system [3] and backstepping-based control system [4] were used to deal with an underactuated helicopter system. However, controller design of an underactuated helicopter system with uncertainties is an open problem.

In this paper, by extending a combined adaptive and non-adaptive attitude control method of 2-input 2-output helicopter experimental system [5,7], a combined attitude control for a three-degree of freedom helicopter experimental system that has 2-input and 3-output [3] is designed. The combined design method is facilitated by adaptive sliding mode control method and some nonlinear control methods to the uncertain underactuated system. Stability of the control system is guaranteed by Lyapunov function based proof. Comparing with PD-based control system [3], the proposed controller shows a desired control result by simulations.

2. MODEL AND PROBLEM STATEMENT

We introduce the model of three-degree of freedom helicopter experimental system with 2-input and 3-output. The photo of the experimental system is shown in Fig. 1.

[1] Partially supported by the JSPS Grant.

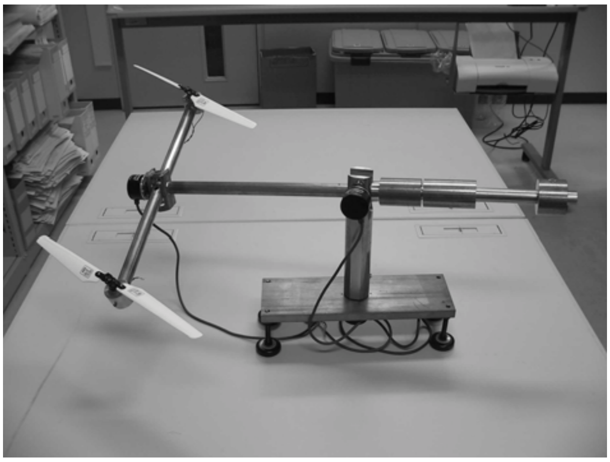

Fig. 1. Helicopter experimental system

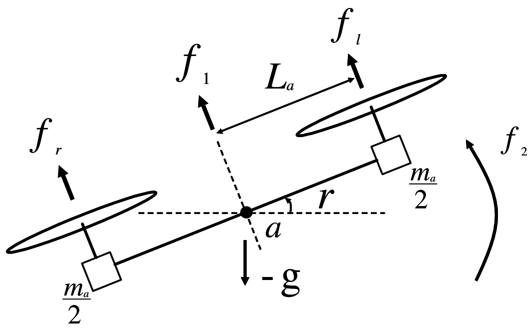

Fig. 2. Relationships of forces on role angle

The experimental system is 2-input 3-output system which attaches the two motors for turning the two rotors, and detects pitch angle, roll angle and yaw angle by rotary encoders. The model on roll angle direction is as follows (Fig. 2).

$$\tilde{f}_1 = f_r + f_l \tag{1}$$
$$\tilde{f}_2 = L_a(f_l - f_r) \tag{2}$$

$$f_r = Ak^2u_r^2 = Ak^2u_1 \tag{3}$$
$$f_l = Ak^2u_l^2 = Ak^2u_2 \tag{4}$$

$$\omega_r = ku_r \tag{5}$$
$$\omega_l = ku_l \tag{6}$$
$$f_r = A\omega_r^2 \tag{7}$$
$$f_l = A\omega_l^2 \tag{8}$$

where, $\tilde{f}_1$ is a resultant force of f_r and f_l, $\tilde{f}_2$ is a moment of the roll direction, and

$\omega_r : Angular\ speed\ of\ right\ rotor$

$\omega_l : Angular\ speed\ of\ left\ rotor$

$k\ : Coefficient\ between\ volt.\ and\ angular\ speed$

$f_r : Lift\ force\ of\ right\ rotor$

$f_l : Lift\ force\ of\ left\ rotor$

$A : Coefficient\ based\ on\ shape\ of\ the\ rotor$

Based on the relationship, we have that

$$\begin{aligned} &I_r\ddot{r} + D_r\dot{r} = L_aAk^2(u_2 - u_1) \\ &I_p\ddot{p} + D_p\dot{p} + mL_gg\sin p \\ &\quad = L_mAk^2(u_1 + u_2)\cos r \\ &I_y\ddot{y} + D_y\dot{y} = L_mAk^2(u_1 + u_2)\sin r \end{aligned} \tag{9}$$

That is,

$$\begin{aligned} \ddot{r} &= -\frac{a_2}{a_1}\dot{r} + \frac{1}{a_1}(u_2 - u_1) \\ \ddot{p} &= -\frac{b_2}{b_1}\dot{p} - \frac{b_3}{b_1}\sin p + \frac{1}{b_1}(u_1 + u_2)\cos r \\ \ddot{y} &= -\frac{c_2}{c_1}\dot{y} + \frac{1}{c_1}(u_1 + u_2)\sin r \end{aligned} \tag{10}$$

$$\begin{aligned} a_1 &= \frac{I_r}{L_aAk^2}, b_1 = \frac{I_p}{L_mAk^2} \\ a_2 &= \frac{D_r}{L_aAk^2}, b_2 = \frac{D_p}{L_mAk^2} \\ c_1 &= \frac{I_y}{L_mAk^2}, b_3 = \frac{mL_gg}{L_mAk^2} \\ c_2 &= \frac{D_y}{L_mAk^2} \end{aligned} \tag{11}$$

The detailed dynamic equations of the angles are given in Appendix. For the simplification, we rearrange motion equations as follows.

$$\begin{aligned} \ddot{r} &= -d_2\dot{r} + d_1(u_2 - u_1) \\ \ddot{p} &= -e_2\dot{p} - e_3\sin p + e_1(u_1 + u_2)\cos r \\ \ddot{y} &= -f_2\dot{y} + f_1(u_1 + u_2)\sin r \end{aligned} \tag{12}$$

$$\begin{aligned} d_1 &= \frac{1}{a_1}, e_1 = \frac{1}{b_1} \\ d_2 &= \frac{a_2}{a_1}, e_2 = \frac{b_2}{b_1} \\ f_1 &= \frac{1}{c_1}, e_3 = \frac{b_3}{b_1} \\ f_2 &= \frac{c_2}{c_1} \end{aligned} \tag{13}$$

where the helicopter system is controlled by u_1 and u_2.

Further, by using the result in [5,7], we obtain the following state equation of the helicopter system.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), t\big) + \boldsymbol{g}\big(\boldsymbol{x}(t), t\big)\boldsymbol{u}(t) \tag{14}$$

where

$$\boldsymbol{f}\big(\boldsymbol{x}(t), t\big) = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ -d_2x_4 \\ -e_2x_5 - e_3\sin x_2 \\ -f_2x_6 \end{pmatrix}$$

$$\mathbf{x(t)} = [x_1, \cdots, x_6]^T = [r, p, y, \dot{r}, \dot{p}, \dot{y}]^T$$

$$\boldsymbol{g}\big(\boldsymbol{x}(t), t\big) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -d_1 & 0 & 0 \\ 0 & e_1 \cos x_1 & 0 \\ 0 & 0 & f_1 \sin x_1 \end{pmatrix}$$

$$\boldsymbol{u}(t) = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad (15)$$

where $\boldsymbol{u}(t)$ is the equivalent control input. In this case, the system has three inputs z_1, z_2 and z_3. Concerning with the real experimental system, there exists modeling error [5, 7]. For this reason, uncertainties are considered. Define $\boldsymbol{f}\big(\boldsymbol{x}(t), t\big)$ and $\boldsymbol{g}\big(\boldsymbol{x}(t), t\big)$ in (14) as

$$\begin{aligned} \boldsymbol{f}\big(\boldsymbol{x}(t), t\big) &= \bar{\boldsymbol{f}}\big(\boldsymbol{x}(t), t\big) + \Delta\boldsymbol{f}\big(\boldsymbol{x}(t), t\big) \\ \boldsymbol{g}\big(\boldsymbol{x}(t), t\big) &= \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big) + \Delta\boldsymbol{g}\big(\boldsymbol{x}(t), t\big) \end{aligned} \quad (16)$$

Substituting these variables into (14), then

$$\begin{aligned} \dot{\boldsymbol{x}}(t) = & \bar{\boldsymbol{f}}\big(\boldsymbol{x}(t), t\big) + \Delta\boldsymbol{f}\big(\boldsymbol{x}(t), t\big) \\ & + \Delta\boldsymbol{g}\big(\boldsymbol{x}(t), t\big)\boldsymbol{u}(t) + \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\boldsymbol{u}(t) \end{aligned} \quad (17)$$

As a result, we represent the state equation as follows. For brevity, we omitted the detailed calculation (see [5,7]).

$$\begin{aligned} \dot{\boldsymbol{x}}(t) = & \bar{\boldsymbol{f}}\big(\boldsymbol{x}(t), t\big) + \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\boldsymbol{h}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big) \\ & + \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t), t\big) + \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\boldsymbol{u}(t) \end{aligned} \quad (18)$$

where the structure of uncertainty $\boldsymbol{h}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big)$ is known, but parameters are unknown, i.e. uncertainty of an indefinite part. Moreover, $\Delta\boldsymbol{h}\big(\boldsymbol{x}(t), t\big)$ is called uncertainty of an indefinite part, and is an unknown model. However, it is referred to as:

$$\big\|\Delta\boldsymbol{h}\big(\boldsymbol{x}(t), t\big)\big\| \leq \eta\big(\boldsymbol{x}(t), t\big) \quad (19)$$

and the upper-bound value function $\eta\big(\boldsymbol{x}(t), t\big)$ is considered as bounded and known. In order for $\Delta\boldsymbol{f}\big(\boldsymbol{x}(t), t\big) + \Delta\boldsymbol{g}\big(\boldsymbol{x}(t), t\big)\boldsymbol{u}(t)$ to fulfill matching conditions for sliding mode control, it is necessary to calculate $\boldsymbol{h}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big)$, defined by:

$$\begin{aligned} & \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\boldsymbol{h}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big) \\ & = \Delta\boldsymbol{f}\big(\boldsymbol{x}(t), t\big) + \Delta\boldsymbol{g}\big(\boldsymbol{x}(t), t\big)\boldsymbol{u}(t) \end{aligned} \quad (20)$$

The objective of this paper is to design a combined attitude controller (u_1 and u_2) so that the controller can control the three system outputs, namely, pitch angle p, roll angle r and yaw angle y. The deign method has two steps, that is, to design $\boldsymbol{u}(t) = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, and then, to realize the control $\boldsymbol{u}(t)$ by u_1 and u_2.

3. COMBINED ATTITUDE CONTROLLER DESIGN

In this section, a combined attitude controller is designed and the control system stability proof is also given.

The proposed combined controller includes a linear feedback controller, a relay controller, a nonlinear controller for the uncertainty of a definite part, a nonlinear controller for the uncertainty of an indefinite part, and a controller for a set point variation. Namely,

$$\begin{aligned} \boldsymbol{u}(t) = & \boldsymbol{u}_{eq}(t) + \boldsymbol{u}_{lf}(t) + \boldsymbol{u}_{nl}(t) \\ & + \boldsymbol{u}_{ad}(t) + \boldsymbol{u}_{\eta}(t) + \boldsymbol{u}_{r}(t) \end{aligned} \quad (21)$$

The detailed structure of these controllers is given as follows.

The equivalent controller
The equivalent controller is expressed by the following equation.

$$\boldsymbol{u}_{eq}(t) = -\Big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\Big)^{-1}\boldsymbol{S}\bar{\boldsymbol{f}}\big(\boldsymbol{x}(t), t\big) \quad (22)$$

where, a switching surface is designed as

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \boldsymbol{S}\Big(\boldsymbol{x}(t) - \boldsymbol{y}_r(t)\Big) \quad (23)$$

$$\boldsymbol{S} = \begin{pmatrix} \iota_1 & 0 & 1 & 0 \\ 0 & \iota_2 & 0 & 1 \end{pmatrix} \quad (24)$$

and $\boldsymbol{y}_r(t)$ is taken as the first order filter output of a step response.

Linear feedback controller
The linear feedback controller is expressed by the following equation.

$$\boldsymbol{u}_{lf}(t) = -\Big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\Big)^{-1}\boldsymbol{K}\boldsymbol{\sigma} \quad (25)$$

where, constant matrix $\boldsymbol{K}$ is a positive definite.

Relay controller
The relay controller is expressed by the following equation.

$$\boldsymbol{u}_{nl}(t) = -\Big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t), t\big)\Big)^{-1}\boldsymbol{\kappa}\ \boldsymbol{\Lambda}(\boldsymbol{\sigma}) \quad (26)$$

$$\boldsymbol{\Lambda}(\boldsymbol{\sigma}) = \begin{pmatrix} \mathrm{sgn}\ (\sigma_1) \\ \mathrm{sgn}\ (\sigma_2) \\ \mathrm{sgn}\ (\sigma_3) \end{pmatrix} \quad (27)$$

$$\mathrm{sgn}\ (x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases} \quad (28)$$

where, $\kappa > 0$.

Controller for definite part uncertainty
Based on the result in [5], we have

$$\boldsymbol{h}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big) = \boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\boldsymbol{\theta}$$

where $\boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)$ is a 3×7 matrix, and

$$\boldsymbol{\theta} = \Big(\, \theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6\ \theta_7 \,\Big)^T$$

Because $\boldsymbol{\theta}$ is unknown, it is calculated using on-line identification, where, the estimate of $\boldsymbol{\theta}$ is set to:

$$\hat{\boldsymbol{\theta}}(t) = \Big(\, \hat\theta_1\ \hat\theta_2\ \hat\theta_3\ \hat\theta_4\ \hat\theta_5\ \hat\theta_6\ \hat\theta_7 \,\Big)^T$$

As a result, the control input to compensate the right hand side 2nd term of (18), which is the term of a presumed error, is designed as follows.

$$\begin{aligned} \boldsymbol{u}_{ad}(t) &= -\hat{\boldsymbol{h}}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big) \\ &= -\boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\hat{\boldsymbol{\theta}}(t) \end{aligned} \qquad (29)$$

where $\hat{\boldsymbol{\theta}}(t)$ is obtained by

$$\dot{\hat{\boldsymbol{\theta}}}(t) = \boldsymbol{\Gamma}\Big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\Big)^T\boldsymbol{\sigma} \qquad (30)$$

Controller for indefinite part uncertainty
The control input to the unknown model is designed as

$$\boldsymbol{u}_\eta(t) = -\eta\big(\boldsymbol{x}(t),t\big)\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big) \qquad (31)$$

where

$$\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big) = \begin{pmatrix} \mathrm{sgn}\ (\tau_1) \\ \mathrm{sgn}\ (\tau_2) \\ \mathrm{sgn}\ (\tau_3) \end{pmatrix} \qquad (32)$$

$$\begin{aligned} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} &= \Big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^T\boldsymbol{\sigma} \\ &= \Big(\boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^T \end{aligned} \qquad (33)$$

Controller for set-point variation
$\boldsymbol{u}_r(t)$ is considered for a set point varying as follows.

$$\begin{aligned} \boldsymbol{u}_r(t) &= \begin{pmatrix} u_{r1} \\ u_{r2} \end{pmatrix} \\ &= \Big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^{-1}\boldsymbol{S}\dot{\boldsymbol{y}}_r(t) \end{aligned} \qquad (34)$$

where $\dot{\boldsymbol{y}}_r(t)$ is bounded.

In the following, the explanation on the possibility for control of the underactuated system is shown. That is, to realize the control $\boldsymbol{u}(t)$ by u_1 and u_2 is explained. From (12) and (15), define that

$$\begin{aligned} F_r &= -d_2x_4 + d_1(u_2 - u_1) \\ F_p &= -e_2x_5 - e_3\sin x_2 + e_1(u_1+u_2)\cos x_1 \\ F_y &= -f_2x_6 + f_1(u_1+u_2)\sin x_1 \end{aligned} \qquad (35)$$

Then, we have that

$$\ddot r = F_r \qquad (36)$$

$$\ddot p = F_p \qquad (37)$$

$$\ddot y = F_y \qquad (38)$$

Now, we design the desired F_p and F_y as follows.

$$F_p^* = z_2 \qquad (39)$$

$$F_y^* = z_3 \qquad (40)$$

where, z_2 and z_3 are given in (15) and (21). From (35), (39) and (40), we obtain the desired role angle as

$$x_1^* = \tan^{-1}\left(\frac{e_1}{f_1}\frac{z_3 + f_2x_6}{z_2 + e_2x_5 + e_3\sin x_2}\right) \qquad (41)$$

and

$$u_1 + u_2 = \frac{z_1 + e_2x_5 + e_3\sin x_2}{e_1\cos x_1} \qquad (42)$$

$$u_2 - u_1 = \frac{z_1 + d_2x_4}{d_1} \qquad (43)$$

where, but the reference input for role angle is x_1^*. In the following, a brief proof of control system stability is shown. Let V be Lyapunov function,

$$\begin{aligned} V &= \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\sigma} + \frac{1}{2}\Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big)^T\boldsymbol{\Gamma}^{-1}\Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big) \\ &= \frac{1}{2}\left(\, \sigma_1\ \sigma_2\ \sigma_3 \,\right)\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \\ &\quad + \frac{1}{2}\left(\, \hat\theta_1 - \theta_1\ \hat\theta_2 - \theta_2\ \cdots\ \hat\theta_7 - \theta_7 \,\right)\boldsymbol{\Gamma}^{-1} \\ &\quad \begin{pmatrix} \hat\theta_1 - \theta_1 \\ \hat\theta_2 - \theta_2 \\ \cdots \\ \hat\theta_7 - \theta_7 \end{pmatrix} \end{aligned}$$

where, $\boldsymbol{\Gamma}$ is positive define, and

$$\boldsymbol{\Gamma} = \begin{pmatrix} \gamma_1 & \cdots & & \mathbf{0} \\ & \gamma_2 & & \\ & & \cdots & \\ \mathbf{0} & & & \gamma_7 \end{pmatrix} \qquad (44)$$

Then, we have that

$$V = \frac{1}{2}\Big(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\Big) + \frac{1}{2}\sum_{n=1}^{7}\frac{1}{a_n}\Big(\hat\theta_n - \theta_n\Big)^2 \qquad (45)$$

It is obvious that

$$V(x) > 0 \qquad (46)$$

From (23),

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= \boldsymbol{S}\Big(\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{y}}_r(t)\Big)\\
&= \boldsymbol{S}\Big(\bar{\boldsymbol{f}}\big(\boldsymbol{x}(t),t\big) + \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{h}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\\
&\quad + \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big) + \bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{u}(t) - \dot{\boldsymbol{y}}_r(t)\Big)\\
&= \boldsymbol{S}\bar{\boldsymbol{f}}\big(\boldsymbol{x}(t),t\big) + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{h}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\\
&\quad + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big)\\
&\quad + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{u}(t) - \boldsymbol{S}\dot{\boldsymbol{y}}_r(t) \qquad (47)\end{aligned}$$

Further, from (21) we have

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= \boldsymbol{S}\bar{\boldsymbol{f}}\big(\boldsymbol{x}(t),t\big) + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{h}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\\
&\quad + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big) + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\\
&\quad \Big(\boldsymbol{u}_{eq}(t) + \boldsymbol{u}_{lf}(t) + \boldsymbol{u}_{nl}(t) + \boldsymbol{u}_{ad}(t)\\
&\quad + \boldsymbol{u}_{\eta}(t) + \boldsymbol{u}_r(t)\Big) - \boldsymbol{S}\dot{\boldsymbol{y}}_r(t)\\
&= \boldsymbol{S}\bar{\boldsymbol{f}}\big(\boldsymbol{x}(t),t\big) + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{h}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\\
&\quad + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big) + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\\
&\quad \Big(-\big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\big)^{-1}\boldsymbol{S}\bar{\boldsymbol{f}}\big(\boldsymbol{x}(t),t\big)\\
&\quad - \big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\big)^{-1}\boldsymbol{K}\boldsymbol{\sigma}\\
&\quad - \big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\big)^{-1}\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma}) - \hat{\boldsymbol{h}}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\\
&\quad - \eta\big(\boldsymbol{x}(t),t\big)\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big) + \big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\big)^{-1}\\
&\quad \boldsymbol{S}\dot{\boldsymbol{y}}_r(t)\Big) - \boldsymbol{S}\dot{\boldsymbol{y}}_r(t)\\
&= -\boldsymbol{K}\boldsymbol{\sigma} - \kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma}) - \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\\
&\quad \Big(\hat{\boldsymbol{h}}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big) - \boldsymbol{h}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\Big)\\
&\quad - \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\eta\big(\boldsymbol{x}(t),t\big)\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big)\\
&\quad + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big)\\
&= -\boldsymbol{K}\boldsymbol{\sigma} - \kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma}) - \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\\
&\quad \boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big)\\
&\quad - \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\eta\big(\boldsymbol{x}(t),t\big)\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big)\\
&\quad + \boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big) \qquad (48)\end{aligned}$$

As a result, the derivative of V is

$$\begin{aligned}\dot{V} &= \boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} + \Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big)^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\theta}}}(t)\\
&= -\boldsymbol{\sigma}^T\boldsymbol{K}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma})\\
&\quad - \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big)\\
&\quad - \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\eta\big(\boldsymbol{x}(t),t\big)\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big)\\
&\quad + \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big) + \Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big)^T\\
&\quad \Big(\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\Big)^T\boldsymbol{\sigma}\\
&= -\boldsymbol{\sigma}^T\boldsymbol{K}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma})\\
&\quad - \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big)\\
&\quad - \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\eta\big(\boldsymbol{x}(t),t\big)\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big)\\
&\quad + \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big)\\
&\quad + \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\boldsymbol{\Phi}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),t\big)\Big(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\Big)\\
&= -\boldsymbol{\sigma}^T\boldsymbol{K}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma})\\
&\quad - \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\eta\big(\boldsymbol{x}(t),t\big)\boldsymbol{\chi}\big(\boldsymbol{x}(t),t\big)\\
&\quad + \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big)\\
&= -\boldsymbol{\sigma}^T\boldsymbol{K}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma}) - \left\|\Big(\boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^T\right\|\\
&\quad \eta\big(\boldsymbol{x}(t),t\big) + \boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big) \qquad (49)\end{aligned}$$

where the Schwarz inequality is used, namely,

$$\begin{aligned}\dot{V} &\le -\boldsymbol{\sigma}^T\boldsymbol{K}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma})\\
&\quad - \left\|\Big(\boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^T\right\|\eta\big(\boldsymbol{x}(t),t\big)\\
&\quad + \left\|\Big(\boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^T\right\|\cdot\|\,\Delta\boldsymbol{h}\big(\boldsymbol{x}(t),t\big)\,\|\\
&\le -\boldsymbol{\sigma}^T\boldsymbol{K}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma})\\
&\quad - \left\|\Big(\boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^T\right\|\eta\big(\boldsymbol{x}(t),t\big)\\
&\quad + \left\|\Big(\boldsymbol{\sigma}^T\boldsymbol{S}\bar{\boldsymbol{g}}\big(\boldsymbol{x}(t),t\big)\Big)^T\right\|\eta\big(\boldsymbol{x}(t),t\big)\\
&\le -\boldsymbol{\sigma}^T\boldsymbol{K}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\kappa\boldsymbol{\Lambda}(\boldsymbol{\sigma}) \qquad (50)\\
&= -\begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}\boldsymbol{K}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}\\
&\quad - \begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}\kappa\begin{pmatrix}\mathrm{sgn}\ (\sigma_1)\\ \mathrm{sgn}\ (\sigma_2)\\ \mathrm{sgn}\ (\sigma_3)\end{pmatrix} \qquad (51)\\
&= -\begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}\begin{pmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}\\
&\quad - d\begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}\begin{pmatrix}\mathrm{sgn}\ (\sigma_1)\\ \mathrm{sgn}\ (\sigma_2)\\ \mathrm{sgn}\ (\sigma_3)\end{pmatrix} \qquad (52)\\
&= -(a\sigma_1^2 + b\sigma_2^2 + c\sigma_3^2)\\
&\quad - d(\|\sigma_1\| + \|\sigma_2\| + \|\sigma_3\|) \qquad (53)\end{aligned}$$

where $\boldsymbol{K} = \begin{pmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{pmatrix}$ is positive define. Since $\dot{V}$ becomes negative definite for $\boldsymbol{\sigma}$, the stability of the system with the equivalent control (21) is ensured. So far, it is not clear if the control using u_1 and u_2 can also guarantee the system stability, where, u_1 and u_2 are obtained by (42) and (43), From the relationship shown in (41), (42) and (43), based on the above proof, u_1, u_2 and x_1^* are bounded provided that $z_2 + e_2x_5 + e_3\sin x_2 \neq 0$ and $x_1 \neq \frac{\pi}{2}$. The, we can control the helicopter system by using u_1 and u_2 [3,4]. In the real control, for the obtained u_1 and u_2, by using the relationship shown in (3) $\sim$ (6), right rotor and left rotor are controlled.

4. SIMULATION

Simulations by using the design method and the PD controller in [3] are performed. In the simulation, as an example, we change the desired reference value of yaw angle at 30[Sec], the desired control result is obtained (see Fig. 3). In this case, using PD controller can also obtain satisfied result, where we omitted it. However, when uncertainties exist, the proposed controller shows a desired control result. In the simulation, we change the physical parameters of the system (10) as the above mentioned uncertainties, where $a_1 = 16.4059$, $a_2 = 1.02117$, $b_1 = 12.4411$, $b_2 = 1.02117$, $b_3 = 26.1264$, $c_1 = 12.4411$, $c_2 = 1.83941$. The other related values are shown in Tables 1 and 2.

I_r	0.06[kg·m^2]
I_p	0.07[kg·m^2]
I_y	0.07[kg·m^2]
L_a	0.195[m]
L_m	0.3[m]
$m_a/2$	0.15[kg]

Table 1. Parameters of the experimental system

	Role angle	Pitch angle	Yaw angle
Initial value [rad]	0	1.13	0
Reference 1[rad]	x_1^*	1.57	0
Reference 2[rad]	x_1^*	1.57	0.62

Table 2. Values of angles

The design parameters of the PD controller are given in Table 3.

	Role angle	Pitch angle	Yaw angle
K_p	1.8	3.5	1.5
K_d	4.4	9.2	28.8

Table 3. Parameters of the PD controller

Simulation results by changing the parameters are shown in Figs. 4 and 5. The desired control result is obtained by using the proposed controller. In the two simulation, the initial values are same, but the desired reference x_1^* is different based on the different control algorithm.

5. CONCLUSION

In this paper, combined attitude control scheme for an underactuated helicopter experimental system is designed. Comparing with the existed result by PD controller, the proposed method is effective for the existence of uncertainties.

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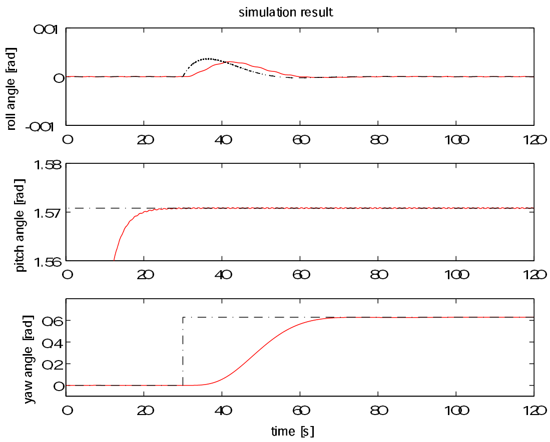

Fig. 3. System outputs by using the proposed controller (angles)

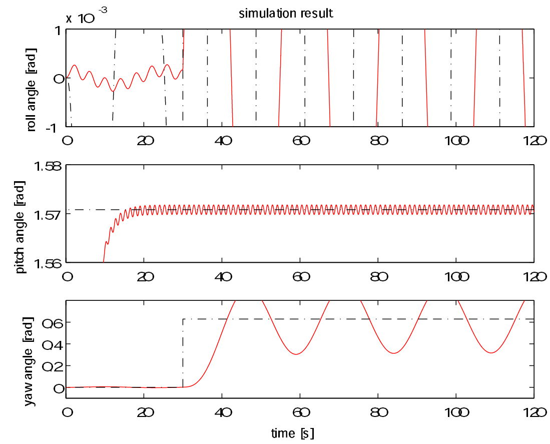

Fig. 4. System output using PD controller

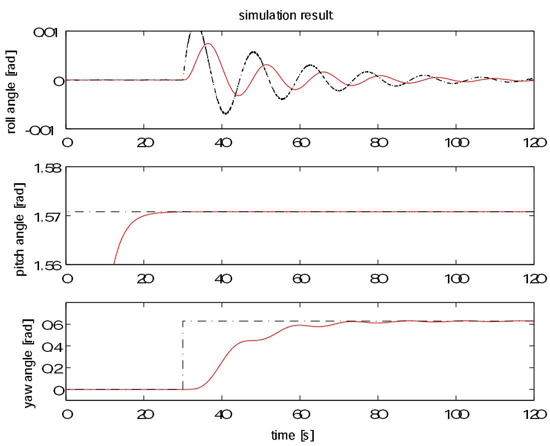

Fig. 5. System output using the proposed controller

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Appendix

Based on Figs. 6 and 7, the dynamic equations on roll angle, pitch angle and yaw angle of the helicopter are shown as follows [6].

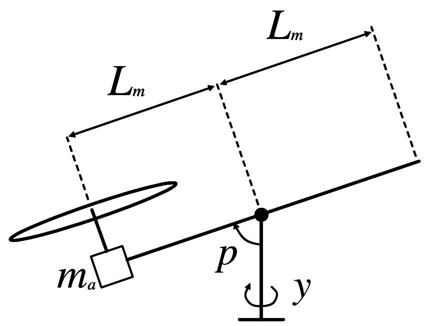

Fig. 6. Model on pitch angle

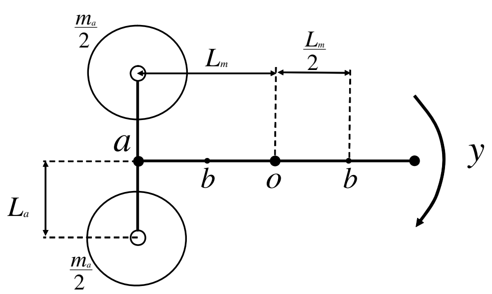

Fig. 7. Model on yaw angle

$$I_r\ddot{r} + D_r\dot{r} = \tilde{f}_2 \tag{54}$$

$$I_p\ddot{p} + D_p\dot{p} + m_a L_m g \sin p = L_m \tilde{f}_1 \cos r \tag{55}$$

$$I_y\ddot{y} + D_y\dot{y} = L_m \tilde{f}_1 \sin r \tag{56}$$

$$I_r = m_a L_a^2 \tag{57}$$

$$I_p = I_{ap} + m_a L_m^2 + 2I_b + \frac{1}{2} m_b L_m^2 \tag{58}$$

$$I_y = I_{ap} + m_a L_m^2 + 2I_b + \frac{1}{2} m_b (L_m \sin p)^2 \tag{59}$$

where

$\frac{m_a}{2}$: *Weight of motor*
g : *Gravity acceleration*
r : *Role angle*
p : *Pitch angle*
y : *Yaw angle*
I_r : *Moment on direction of role angle*
I_p : *Moment on direction of pitch angle*
I_y : *Moment on direction of yaw angle*
D_r : *Friction coef. on direction of role angle*
D_p : *Friction coef. on direction of pitch angle*
D_y : *Friction coef. on direction of yaw angle*
L_a : *Distance from role angle axis to motor*
L_m : *Distance from pitch angle axis to motor*