Nonlinear pole assignment control of state dependent parameter models with time delays

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Abstract: This paper considers pole assignment control of nonlinear dynamic systems described by State Dependent Parameter (SDP) models. The approach follows from earlier research into linear Proportional-Integral-Plus (PIP) methods but, in SDP system control, the control coefficients are updated at each sampling instant on the basis of the latest SDP relationships. Alternatively, algebraic solutions can be derived off-line to yield a practically useful control algorithm that is relatively straightforward to implement on a digital computer, requiring only the storage of lagged system variables, coupled with straightforward arithmetic expressions in the control software. Although the analysis is limited to the case when the open-loop system has no zeros, time delays are handled automatically. The paper shows that the closed-loop system reduces to a linear transfer function with the specified (design) poles. Hence, assuming pole assignability at each sample, global stability of the nonlinear system is guaranteed at the design stage. The associated conditions for pole assignability are stated.

Keywords: control system design; nonlinear systems; state dependent parameters

1. INTRODUCTION

Recent papers have shown the practical utility of an approach to nonlinear control based on non-minimal state variable feedback [Stables and Taylor, 2006, Taylor et al., 2007b]. Here, the nonlinear system is modelled using a quasi-linear State Dependent Parameter (SDP) structure, in which the parameters vary as functions of the state variables [Young et al., 2001].

Using an analogy with linear Proportional-Integral-Plus (PIP) methods [Young et al., 1987, Taylor et al., 2000], state variable feedback control can subsequently be implemented directly from the measured input and output signals, without resort to the design of a deterministic state reconstructor (observer) or a stochastic Kalman filter. In the nonlinear case, this formulation is used to design a SDP/PIP control law at each sampling instant using linear methods such as pole assignment or (suboptimal) Linear Quadratic (LQ) design.

The closely related State Dependent Riccati Equation (SDRE) approach to nonlinear control is a growing area of both theoretical and practical research; see e.g. Banks et al. [2007] and the references therein. Here, a direct parameterisation is used to transform the nonlinear system into a linear structure having state-dependent parameters. Using this model, the SDRE method mimics the LQ regulator for linear systems but yields a state variable feedback control algorithm with time varying coefficients [Banks and Mhana, 1992].

In the discrete-time case, as considered in the present paper, the ubiquitous algebraic Riccati equation is solved at each sampling interval on an assumption of point-wise controllability [e.g. Dutka et al., 2005, Taylor et al., 2007b]. However, while some theoretical advances have been made regarding the asymptotic stability of the SDRE approach, the conditions obtained can be difficult to check and/or fulfill. Note also that, whilst optimality is theoretically achievable for continuous-time systems, this is not necessarily the case for discrete-time systems.

By contrast, the present paper concentrates on a novel *pole assignment* approach that guarantees the specified closed-loop response (and hence stability) at the design stage. The approach is an extension of the pole assignment algorithm developed by Stables and Taylor [2006]. However, whilst the latter was limited to unity time delay systems ($\tau = 1$), utilising a Smith Predictor to handle longer time delays, the present paper generalises these results to the $\tau \geq 1$ case. The paper shows that the closed-loop system reduces to a linear transfer function with the expected design poles (Section 3). The proposed control algorithm is illustrated with a 3rd order worked example based on 3 samples pure time delay (Section 4).

2. BACKGROUND

Consider the deterministic form of the SDP model:

$$y_k = \mathbf{w}_k^{\mathrm{T}} \mathbf{p}_k \qquad (1)$$

where $\mathbf{w}_k^{\mathrm{T}}$ is a vector of lagged input and output variables and $\mathbf{p}_k$ is a vector of SDP parameters,

$$\mathbf{w}_k^{\mathrm{T}} = [\, -y_{k-1} \;\cdots\; -y_{k-n} \;\; u_{k-\tau} \;\cdots\; u_{k-\tau-m+1} \,]$$
$$\mathbf{p}_k^{\mathrm{T}} = [\, a_1\{\boldsymbol{\chi}_k\} \;\cdots\; a_n\{\boldsymbol{\chi}_k\} \;\; b_1\{\boldsymbol{\chi}_k\} \;\cdots\; b_m\{\boldsymbol{\chi}_k\} \,]$$

Here y_k is the output and u_k the control input, while $a_i\{\boldsymbol{\chi}_k\}$ and $b_i\{\boldsymbol{\chi}_k\}$ are state dependent parameters. The latter are assumed to be functions of a state vector $\boldsymbol{\chi}_k$ consisting of measured variables. Finally, n and m are integers representing the number of SDP parameters associated with the output and input respectively; and τ is the pure time (transport) delay of the system where, for the purposes of discrete-time control system design, $\tau \geq 1$.

Numerous recent papers by the third author and colleagues have described an approach for the identification and estimation of such models; and have illustrated their application to a wide range of practical examples: see e.g. Young et al. [2001], Taylor et al. [2007a] and the references therein.

2.1 Subset model

For basic SDP/PIP control system design, it is usually sufficient to limit the model (1) to the case $\boldsymbol{\chi}_k = \mathbf{w}_k$. In fact, the analysis below concentrates on a further subset of the entire class of SDP models, represented by the following difference equation,

$$\begin{aligned} y_k = &- a_1\{\boldsymbol{\chi}_k\} y_{k-1} - a_2\{\boldsymbol{\chi}_k\} y_{k-2} \ldots \\ &- a_n\{\boldsymbol{\chi}_k\} y_{k-n} + b_\tau\{\boldsymbol{\chi}_k\} u_{k-\tau} \end{aligned} \tag{2}$$

and equivalent pseudo– transfer function representation,

$$y_k = \frac{b_\tau\{\boldsymbol{\chi}_k\} z^{-\tau}}{A\{\boldsymbol{\chi}_k, z^{-1}\}} u_k$$

where $A\left(\boldsymbol{\chi}_k, z^{-1}\right) = 1 + a_1\{\boldsymbol{\chi}_k\} z^{-1} + \ldots + a_n\{\boldsymbol{\chi}_k\} z^{-n}$ is an appropriately defined state dependent polynomial, in which z^{-1} is the backward shift operator, i.e. $z^{-i} y_k = y_{k-i}$. This model encompass a wide range of nonlinear structures. For example, it has proven particularly useful for the control of ventilation rate [Stables and Taylor, 2006] and hydraulically operated robot arms [Taylor et al., 2007b], where the latter are represented as integrators with time delay and state dependent gain.

The 'input' component of the model is limited to a single element $b_\tau\{\boldsymbol{\chi}_k\}$, i.e. the equivalent pseudo- transfer function would have no zeros. This is because pole assignment methods are concerned with the placement of the closed-loop pole positions only. In fact, assuming no model mismatch, linear PIP design yields a closed-loop transfer function with the same zeros as the open-loop model.

In the linear case, this result never affects the stability of the system, although clearly zeros that lie outside the unit circle on the complex z-plane, may yield non-minimum phase behaviour [Taylor et al., 2000]. However, in the case of nonlinear systems, the influence of such numerator terms is more complex and may lead to an undesirable response for conventional SDP/PIP design [Taylor, 2008].

2.2 State variable feedback design

The non-minimal state space (NMSS) representation of the system (2) is:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{F}_k \mathbf{x}_k + \mathbf{g}_k u_k + \mathbf{d} r_{k+1} \\ y_k &= \mathbf{h} \mathbf{x}_k \end{aligned} \tag{3}$$

where the $n+\tau$ dimensional state vector is:

$$\mathbf{x}_k = [y_k \;\cdots\; y_{k-n+1} \;\; u_{k-1} \;\cdots\; u_{k-\tau+1} \;\; z_k]^T \tag{4}$$

and $z_k = z_{k-1} + [r_k - y_k]$ is the integral-of-error between the reference or command input r_k and the sampled output y_k. As usual for NMSS design, inherent type 1 servomechanism performance is introduced by means of this integral-of-error state [Young et al., 1987]. The state transition matrix $\mathbf{F}_k$ and input vector $\mathbf{g}_k$ at the kth sample, together with the time invariant command $\mathbf{d}$ and observation $\mathbf{h}$ vectors are defined by Taylor et al. [2007b].

The control law takes the usual state variable feedback form,

$$u_k = -\mathbf{c}_k \mathbf{x}_k \tag{5}$$

where the state dependent control gain vector,

$$\mathbf{c}_k = [\, f_{0,k} \;\ldots\; f_{n-1,k} \;\; g_{1,k} \;\ldots\; g_{\tau-1,k} \;\; -k_{I,k} \,] \tag{6}$$

is obtained by either pole assignment or optimisation of a conventional LQ cost function. In either case, Fig. 1 illustrates the SDP/PIP controller in block diagram form, where,

$$F_1\{\boldsymbol{\chi}_k, z^{-1}\} = f_{1,k} z^{-1} + \ldots + f_{n-1,k} z^{-n+1}$$
$$G\{\boldsymbol{\chi}_k, z^{-1}\} = 1 + g_{1,k} z^{-1} + \ldots + g_{\tau-1,k} z^{-\tau+1}$$

while $f_{0,k}$ and $k_{I,k}$ are the time varying (state dependent) proportional and integral gains respectively.

3. NONLINEAR POLE ASSIGNMENT

The closed-loop characteristic polynomial is defined,

$$D(z^{-1}) = 1 + d_1 z^{-1} + \ldots + d_{n+\tau} z^{-(n+\tau)} \tag{7}$$

where d_i are the desired coefficients associated with the closed-loop poles p_i ($i = 1 \ldots n+\tau$). In previous papers on SDP/PIP pole assignment, the coefficients of the closed-loop characteristic polynomial are equated to those of the desired characteristic polynomial $D(z^{-1})$ at each control sample k. Although based on an analogy with linear methods, the required closed-loop behaviour is not guaranteed in the nonlinear case [Taylor, 2008].

By contrast, the present paper uses a modified pole assignment algorithm as follows,

$$\boldsymbol{\Sigma}_k \cdot \mathbf{v}_k = \boldsymbol{\beta}_k \tag{8}$$

where $\mathbf{v}_k$ is a vector of control gains. These differ from $\mathbf{c}_k$ given by equation (6) in terms of the input feedback coefficients, as shown below,

$$\mathbf{v}_k = [\, f_{0,k} \;\ldots\; f_{n-1,k} \;\; \bar{g}_{1,k} \;\ldots\; \bar{g}_{\tau-1,k} \;\; -k_{I,k} \,]^T \tag{9}$$

Here, $\boldsymbol{\beta}_k = [\beta_1, \beta_2, \ldots, \beta_{n+\tau}]^T$ is obtained using the design coefficients d_i as follows,

$$\beta_i = d_i - \left(a_i\{\boldsymbol{\chi}_{k+\tau}\} - a_{i-1}\{\boldsymbol{\chi}_{k+\tau}\}\right) \tag{10}$$

where $a_0\{\boldsymbol{\chi}_{k+\tau}\} = 1$ and $a_i\{\boldsymbol{\chi}_{k+\tau}\} = 0$ for $i > n$. Finally, $\boldsymbol{\Sigma}_k$ is a matrix of dimension $(n+\tau) \times (n+\tau)$, partitioned as follows,

$$\boldsymbol{\Sigma}_k = [\boldsymbol{\Sigma}_{f,k} | \boldsymbol{\Sigma}_{g,k} | \boldsymbol{\Sigma}_{b,k}] \tag{11}$$

where the $(n+\tau) \times 1$ dimensional $\boldsymbol{\Sigma}_{b,k}$ is,

$$\boldsymbol{\Sigma}_{b,k} = \left[\, 0 \;\ldots\; 0 \;\; b_\tau\{\boldsymbol{\chi}_{k+\tau}\} \;\; 0 \;\ldots\; 0 \,\right]^T$$

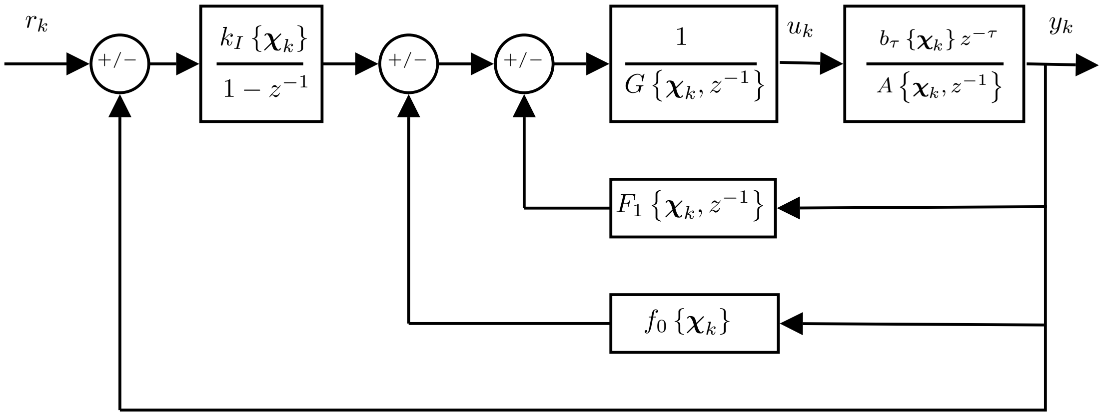

Fig. 1. SDP/PIP control structure.

$$\boldsymbol{\Sigma}_{f,k} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ b_\tau\{\boldsymbol{\chi}_{k+\tau}\} & 0 & \dots & 0 & 0 \\ -b_\tau\{\boldsymbol{\chi}_{k+\tau}\} & b_\tau\{\boldsymbol{\chi}_{k+\tau}\} & & 0 & 0 \\ 0 & -b_\tau\{\boldsymbol{\chi}_{k+\tau}\} & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -b_\tau\{\boldsymbol{\chi}_{k+\tau}\} & b_\tau\{\boldsymbol{\chi}_{k+\tau}\} \\ 0 & 0 & \dots & 0 & -b_\tau\{\boldsymbol{\chi}_{k+\tau}\} \end{bmatrix}$$

Fig. 2. General form of the $(n+\tau) \times n$ $\boldsymbol{\Sigma}_{f,k}$ matrix in equation (11).

$$\boldsymbol{\Sigma}_{g,k} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_1\{\boldsymbol{\chi}_{k+\tau-1}\} - 1 & 1 & \dots & 0 \\ a_2\{\boldsymbol{\chi}_{k+\tau-1}\} - a_1\{\boldsymbol{\chi}_{k+\tau-1}\} & a_1\{\boldsymbol{\chi}_{k+\tau-2}\} - 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n\{\boldsymbol{\chi}_{k+\tau-1}\} & a_n\{\boldsymbol{\chi}_{k+\tau-2}\} - a_{n-1}\{\boldsymbol{\chi}_{k+\tau-2}\} & \dots & 0 \\ 0 & -a_n\{\boldsymbol{\chi}_{k+\tau-2}\} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n\{\boldsymbol{\chi}_{k+1}\} - a_{n-1}\{\boldsymbol{\chi}_{k+1}\} \\ 0 & 0 & \dots & -a_n\{\boldsymbol{\chi}_{k+1}\} \end{bmatrix}$$

Fig. 3. General form of the $(n+\tau) \times (\tau-1)$ $\boldsymbol{\Sigma}_{g,k}$ matrix in equation (11).

with $b_\tau\{\boldsymbol{\chi}_{k+\tau}\}$ on the τth row. The $(n+\tau) \times n$ dimensional $\boldsymbol{\Sigma}_{f,k}$ and $(n+\tau) \times (\tau-1)$ dimensional $\boldsymbol{\Sigma}_{g,k}$ matrices are given by Fig. 2 and Fig. 3 respectively. Note that the first $\tau - 1$ rows of $\boldsymbol{\Sigma}_{f,k}$ and $\boldsymbol{\Sigma}_{b,k}$ are zero. For non-singular $\boldsymbol{\Sigma}_k$ (see Theorem 1 below), it is clear that the coefficients $\mathbf{v}_k$ are computed from equation (8).

In order to obtain the control gain vector $\mathbf{c}_k$ shown by equation (6), the input feedback coefficients $g_{i,k}$ are subsequently scaled as follows,

$$g_{i,k} = \frac{b_\tau\{\boldsymbol{\chi}_{k+\tau-i}\}}{b_\tau\{\boldsymbol{\chi}_{k+\tau}\}} \cdot \bar{g}_{i,k} \quad (i = 1, \dots, \tau - 1) \tag{12}$$

It should be stressed that the time indices of the state dependent parameters in $\boldsymbol{\Sigma}_k$ and $\boldsymbol{\beta}_k$, coupled with the scaling (12), are the key to obtaining an appropriate nonlinear pole assignment control algorithm. In particular, the expected 'design' response, such as deadbeat or a specified response time and overshoot, are only obtained using this approach, as discussed below.

3.1 Pole assignability

A prerequisite of global controllability is that the system $\{\mathbf{F}_k, \mathbf{g}_k, \mathbf{h}, \mathbf{d}\}$ is piecewise controllable at each sample k, with the standard NMSS controllability conditions applying over each sampling period. The NMSS/PIP linear controllability conditions are derived by Young et al. [1987] For the model (2) but with time invariant parameters, these reduce to only $b_\tau \neq 0$, which is equivalent to the standard requirement that the input has some influence on the system output.

Theorem 1 (pole assignability): The nonlinear pole assignment algorithm (8) through to (12), can be solved if and only if $b_\tau\left\{\boldsymbol{\chi}_{k+\tau}\right\} \neq 0, \forall k$

Proof: Equation (8) can be solved if and only if $\boldsymbol{\Sigma}_k$ is non-singular. Although omitted here for brevity, it is straightforward to show that the condition for the non-singularity of the $\boldsymbol{\Sigma}_k$ matrix is $b_\tau\left\{\boldsymbol{\chi}_{k+\tau}\right\} \neq 0, \forall k$ [Taylor, 2008]. When the latter condition holds, equation (12) can also be solved.

It is straightforward to check this condition at every sampling instant. If it fails to hold at the kth sample, the algorithm is instead evaluated for $k-1$. Unfortunately, this effectively leaves the system in linear (fixed gain) mode for a period of time, something that can be undesirable in practice. For this reason, any problem regions of the parameter space need to be investigated by off-line simulation [Taylor et al., 2007b].

3.2 Stability

The analysis below assumes: (i) pole assignability at each sample; and (ii) the input signal generated by (5) is realisable in practice. With regard to the latter point, depending on the value of each SDP, the magnitude of u_k may be unfeasibly large, causing numerical problems in software simulation or practical application. Although space limitations preclude further investigation here, the earlier publications cited above show how practically realisable control algorithms can be obtained. For example, the trajectory of the command input may be chosen to avoid local feasibility problems, if any arise.

Theorem 2 (design poles): The control algorithm (5), solved using equations (8) to (12) and applied to the discrete-time SDP model (2), yields the following linear transfer function in closed-loop:

$$y_k = \frac{1 + d_1 + \ldots + d_{n+\tau}}{1 + d_1 z^{-1} + \ldots + d_{n+\tau} z^{-(n+\tau)}} r_{k-\tau} \qquad (13)$$

Proof: Although again omitted here for brevity, straightforward algebra shows that the above formulation yields equation (13) for nth order SDP systems with time delay $\tau \geq 1$ [Taylor, 2008]. An illustration based on state space analysis is given by Taylor and Chotai [2008].

Corollary (stability): For stable design poles p_i based on equation (7), the control algorithm (5), solved using equations (8) to (12) and applied to the discrete-time SDP model (2), yields a stable closed-loop response. Since equation (13) is a linear transfer function with poles p_i, the proof follows obviously from Theorem 2.

3.3 Implementation

SDP/PIP design requires a forward shift of $\boldsymbol{\chi}_k$ with respect to the model (2), i.e. equations (10) and (12) show that the control algorithm utilises $a_i\left\{\boldsymbol{\chi}_{k+\tau}\right\}$ and $b_\tau\left\{\boldsymbol{\chi}_{k+\tau}\right\}$, respectively, where τ is the time delay. To illustrate, consider a model with $\tau = 3$ and $a_1\left\{\boldsymbol{\chi}_k\right\} = \alpha_1 + \alpha_2 y_{k-3}$, where α_1 and α_2 are coefficients. In this case, it is straightforward to determine equation (10) using the 'latest' measurement of the output, i.e. $a_1\left\{\boldsymbol{\chi}_{k+3}\right\} = \alpha_1 + \alpha_2 y_k$. By contrast, if the SDP model is defined by $a_1\left\{\boldsymbol{\chi}_k\right\} = \alpha_1 + \alpha_2 y_{k-1}$ then an estimate $a_1\left\{\boldsymbol{\chi}_{k+3}\right\} = \alpha_1 + \alpha_2 \hat{y}_{k+2}$ would be required to implement the control algorithm.

Here, one option is to use $\hat{y}_{k+2} = y_{d,k+2}$ where $y_{d,k+2}$ is the design response. In this case, Theorem 2 still holds, albeit with a robustness penalty that can be investigated in simulation and by practical example. This is beyond the scope of the present paper which (in a similar manner to linear PIP design) assumes zero model mismatch at the design stage. However, the publications using SDP/PIP control cited above, have shown that satisfactory control is still possible despite modelling uncertainty and disturbances.

The presence of future values of u_k in the control algorithm represents a more difficult problem. This is most straightforwardly addressed at the model identification stage, by limiting the state dependent variable to be lagged by at least τ samples. Fortunately this makes sense from a practical viewpoint, especially when developing SDP models for a physical system.

Finally, the SDP/PIP control algorithm (5) may be manipulated into an equivalent incremental form, as shown in section 4 below. When combined with the algebraic solution to (8), this form is particularly straightforward to implement, requiring only the storage of lagged system variables, coupled with straightforward arithmetic expressions in the control software.

4. WORKED EXAMPLE

Consider the following SDP model based on equation (2), for which $n = \tau = 3$,

$$\begin{aligned} y_k = & - a_1\left\{\boldsymbol{\chi}_k\right\} y_{k-1} - a_2\left\{\boldsymbol{\chi}_k\right\} y_{k-2} \\ & - a_3\left\{\boldsymbol{\chi}_k\right\} y_{k-3} + b_3\left\{\boldsymbol{\chi}_k\right\} u_{k-3} \end{aligned} \qquad (14)$$

Here, $\mathbf{x}_k = \left[y_k \; y_{k-1} \; y_{k-2} \; u_{k-1} \; u_{k-2} \; z_k\right]^T$ and the pole assignment problem (8) is formulated,

$$\boldsymbol{\Sigma}_k \cdot \begin{bmatrix} f_{0,k} \\ f_{1,k} \\ f_{2,k} \\ \bar{g}_{1,k} \\ \bar{g}_{2,k} \\ k_{I,k} \end{bmatrix} = \begin{bmatrix} d_1 - a_1\left\{\boldsymbol{\chi}_{k+3}\right\} + 1 \\ d_2 - a_2\left\{\boldsymbol{\chi}_{k+3}\right\} + a_1\left\{\boldsymbol{\chi}_{k+3}\right\} \\ d_3 - a_3\left\{\boldsymbol{\chi}_{k+3}\right\} + a_2\left\{\boldsymbol{\chi}_{k+3}\right\} \\ d_4 + a_3\left\{\boldsymbol{\chi}_{k+3}\right\} \\ d_5 \\ d_6 \end{bmatrix} \qquad (15)$$

where $\boldsymbol{\Sigma}_k$ is defined in Fig. 4.

To implement this control system, $\mathbf{v}_k = \boldsymbol{\Sigma}_k^{-1} \boldsymbol{\beta}_k$ could be numerically solved at each sample. For example, the axial fan controller described by Stables and Taylor [2006] utilises Matlab® on-line to find the necessary matrix inversion. However, in the context of the present paper, it is more useful to consider instead the algebraic solution shown below,

$$
\begin{aligned}
\bar{g}_{1,k} &= -a_1\left\{\boldsymbol{\chi}_{k+3}\right\} + 1 + d_1 \\
\bar{g}_{2,k} &= -a_2\left\{\boldsymbol{\chi}_{k+3}\right\} - a_1\left\{\boldsymbol{\chi}_{k+2}\right\}\bar{g}_1 + 1 + d_1 + d_2 \\
f_{0,k} &= \frac{-a_1\left\{\boldsymbol{\chi}_{k+1}\right\}\bar{g}_2 - a_2\left\{\boldsymbol{\chi}_{k+2}\right\}\bar{g}_1}{b_3\left\{\boldsymbol{\chi}_{k+3}\right\}} \\
&\quad + \frac{-a_3\left\{\boldsymbol{\chi}_{k+3}\right\} - d_4 - d_5 - d_6}{b_3\left\{\boldsymbol{\chi}_{k+3}\right\}} \\
f_{1,k} &= \frac{-a_2\left\{\boldsymbol{\chi}_{k+1}\right\}\bar{g}_2 - a_3\left\{\boldsymbol{\chi}_{k+2}\right\}\bar{g}_1 - d_5 - d_6}{b_3\left\{\boldsymbol{\chi}_{k+3}\right\}} \\
f_{2,k} &= \frac{-a_3\left\{\boldsymbol{\chi}_{k+1}\right\}\bar{g}_2 - d_6}{b_3\left\{\boldsymbol{\chi}_{k+3}\right\}} \\
k_{I,k} &= \frac{1 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6}{b_3\left\{\boldsymbol{\chi}_{k+3}\right\}}
\end{aligned}
\tag{16}
$$

The input feedback scaling (12) takes the form,

$$
\begin{aligned}
g_{1,k} &= \frac{b_3\left\{\boldsymbol{\chi}_{k+2}\right\}}{b_3\left\{\boldsymbol{\chi}_{k+3}\right\}} \cdot \bar{g}_{1,k} \\
g_{2,k} &= \frac{b_3\left\{\boldsymbol{\chi}_{k+1}\right\}}{b_3\left\{\boldsymbol{\chi}_{k+3}\right\}} \cdot \bar{g}_{2,k}
\end{aligned}
\tag{17}
$$

and the control algorithm is most obviously implemented in state variable feedback (5) form.

Alternatively, the control algorithm may be manipulated into an incremental form, for which it is helpful to first define $\bar{f}_{i,k} = b_\tau\left\{\boldsymbol{\chi}_{k+\tau}\right\} f_{i,k}$ and $\bar{k}_{I,k} = b_\tau\left\{\boldsymbol{\chi}_{k+\tau}\right\} k_{I,k}$. In this case, substituting $z_k = z_{k-1} + r_k - y_k$ into equation (5) and multiplying through by $b_\tau\left\{\boldsymbol{\chi}_{k+\tau}\right\}$ to simplify the presentation, yields,

$$
\begin{aligned}
& b_\tau\left\{\boldsymbol{\chi}_{k+\tau}\right\} u_k = b_\tau\left\{\boldsymbol{\chi}_{k+\tau-1}\right\} u_{k-1} - \bar{f}_{0,k} y_k \\
& + \bar{f}_{0,k-1} y_{k-1} - \bar{f}_{1,k} y_{k-1} + \bar{f}_{1,k-1} y_{k-2} - \bar{f}_{2,k} y_{k-2} \\
& + \bar{f}_{2,k-1} y_{k-3} - b_3\left\{\boldsymbol{\chi}_{k+3}\right\} g_{1,k} u_{k-1} \\
& + b_3\left\{\boldsymbol{\chi}_{k+2}\right\} g_{1,k-1} u_{k-2} - b_3\left\{\boldsymbol{\chi}_{k+3}\right\} g_{2,k} u_{k-2} \\
& + b_3\left\{\boldsymbol{\chi}_{k+2}\right\} g_{2,k-1} u_{k-3} + \bar{k}_{I,k} r_k - \bar{k}_{I,k} y_k
\end{aligned}
\tag{18}
$$

Since algebraic solutions to equation (8) have been derived off-line to yield equations (16), the incremental form (18) is a practically useful control algorithm that is relatively straightforward to implement on a digital computer, requiring only the storage of lagged system variables, coupled with straightforward arithmetic expressions in the control software, as shown below.

Finally, in order to simulate a numerical example, consider the following SDP functions for the model (14):

$$
\begin{aligned}
a_1\left\{\boldsymbol{\chi}_k\right\} &= 0.2\ y_{k-3} - 0.7 \\
a_2\left\{\boldsymbol{\chi}_k\right\} &= 0.3\ y_{k-4} \cdot u_{k-5} \\
a_3\left\{\boldsymbol{\chi}_k\right\} &= -1.2\ y_{k-5} - 0.8 \\
b_3\left\{\boldsymbol{\chi}_k\right\} &= 0.1\ \arctan\left(u_{k-4}\right) + 2.5
\end{aligned}
$$

Although arbitrarily chosen, these show how each SDP can be defined as linear, trigometric or other functions of the lagged input and output variables. In fact, $a_2\left\{\boldsymbol{\chi}_k\right\}$ is a multiplicative function of both these variables. Substituting these SDP functions into equations (16) yields,

$$
\begin{aligned}
\bar{g}_{1,k} &= -\left(0.2\ y_k - 0.7\right) + 1 + d_1 \\
\bar{g}_{2,k} &= -\left(0.3\ y_{k-1} \cdot u_{k-2}\right) \\
&\quad - \left(0.2\ y_{k-1} - 0.7\right)\bar{g}_1 + 1 + d_1 + d_2 \\
f_{0,k} &= -\frac{\left(0.2\ y_{k-2} - 0.7\right)\bar{g}_2 + \left(0.3\ y_{k-2} \cdot u_{k-3}\right)\bar{g}_1}{0.1\ \arctan\left(u_{k-1}\right) + 2.5} \\
&\quad + \frac{\left(1.2\ y_{k-2} + 0.8\right) - d_4 - d_5 - d_6}{0.1\ \arctan\left(u_{k-1}\right) + 2.5} \\
f_{1,k} &= -\frac{\left(0.3\ y_{k-3} \cdot u_{k-4}\right)\bar{g}_2 - \left(1.2\ y_{k-3} + 0.8\right)\bar{g}_1}{0.1\ \arctan\left(u_{k-1}\right) + 2.5} \\
&\quad + \frac{-d_5 - d_6}{0.1\ \arctan\left(u_{k-1}\right) + 2.5} \\
f_{2,k} &= \frac{\left(1.2\ y_{k-4} + 0.8\right)\bar{g}_2 - d_6}{0.1\ \arctan\left(u_{k-1}\right) + 2.5} \\
k_{I,k} &= \frac{1 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6}{0.1\ \arctan\left(u_{k-1}\right) + 2.5}
\end{aligned}
\tag{19}
$$

Specifying real poles of $p_{1,2} = 0.2$, $p_{3,4} = 0.5 \pm 0.4j$ and $p_{5,6} = 0.7$, where j is the complex number, yields the closed-loop response shown as the thick trace in Fig. 5. As expected, the response is exactly equal to the design response, obtained by simulating a transfer function with these poles in open-loop. In fact, Fig. 5 also shows a dead-beat response, obtained by placing all six closed-loop poles at the origin of the complex z-plane (thin trace). Although the latter design is unrealistic for practical examples with model mismatch, it illustrates Theorem 2 for an extreme case. For the slower of these two designs, Fig. 6 shows the time varying state dependent parameters.

Finally, a linear PIP algorithm based on an operating condition of $y_k = 1$, does not yield a stable solution in response to the command input and poles shown here, highlighting the utility of the nonlinear approach.

5. CONCLUSIONS

This paper has developed a nonlinear pole assignment algorithm for control of systems described by State Dependent Parameter (SDP) models. In particular, it extends earlier research for unity time delay $\tau = 1$ to the more general $\tau \geq 1$ case. Using this formulation, the closed-loop system reduces to a linear transfer function with the specified (design) poles. Hence, assuming pole assignability at each sample and no model mismatch, stability of the nonlinear system is guaranteed for stable design poles.

Although the present paper is limited to the case that the SDP model has no zeros, recourse can be made to a ‘partial linearisation by feedback’ approach to handle any higher order input terms [Shaban, 2006]. Further links with exact linearisation by feedback [Isidori, 1995] and state dependent Riccati equation methods [Banks et al., 2007] are being investigated by the authors.

However, the present paper provides a theoretical motivation for earlier successful practical demonstrators in the area of environmental control and construction robots [Taylor et al., 2007b].

$$\boldsymbol{\Sigma}_k = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_1\{\boldsymbol{\chi}_{k+2}\} - 1 & 1 & 0 \\ b_3\{\boldsymbol{\chi}_{k+3}\} & 0 & 0 & a_2\{\boldsymbol{\chi}_{k+2}\} - a_1\{\boldsymbol{\chi}_{k+2}\} & a_1\{\boldsymbol{\chi}_{k+1}\} - 1 & b_3\{\boldsymbol{\chi}_{k+3}\} \\ -b_3\{\boldsymbol{\chi}_{k+3}\} & b_3\{\boldsymbol{\chi}_{k+3}\} & 0 & a_3\{\boldsymbol{\chi}_{k+2}\} - a_2\{\boldsymbol{\chi}_{k+2}\} & a_2\{\boldsymbol{\chi}_{k+1}\} - a_1\{\boldsymbol{\chi}_{k+1}\} & 0 \\ 0 & -b_3\{\boldsymbol{\chi}_{k+3}\} & b_3\{\boldsymbol{\chi}_{k+3}\} & -a_3\{\boldsymbol{\chi}_{k+2}\} & a_3\{\boldsymbol{\chi}_{k+1}\} - a_2\{\boldsymbol{\chi}_{k+1}\} & 0 \\ 0 & 0 & -b_3\{\boldsymbol{\chi}_{k+3}\} & 0 & -a_3\{\boldsymbol{\chi}_{k+1}\} & 0 \end{bmatrix}$$

Fig. 4. Worked example $\boldsymbol{\Sigma}_k$ matrix in equation (15).

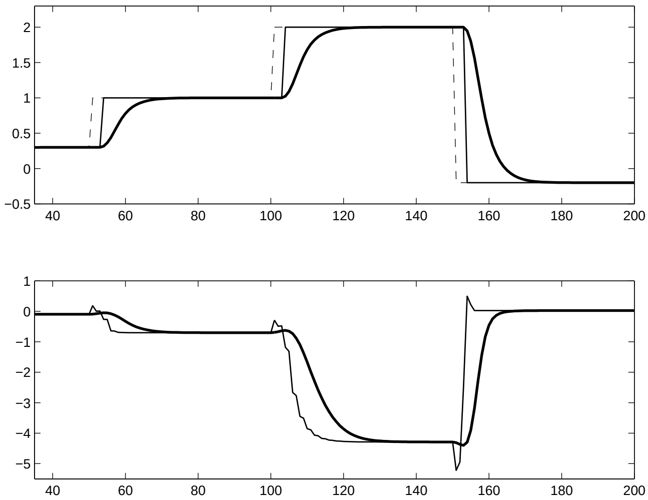

Fig. 5. SDP/PIP control of the example, comparing dead-beat (thin traces) and a response using 'slower' pole positions (thick traces). Upper subplot: output y_k and command input r_k (dashed) plotted against sample number. Lower subplot: control input u_k.

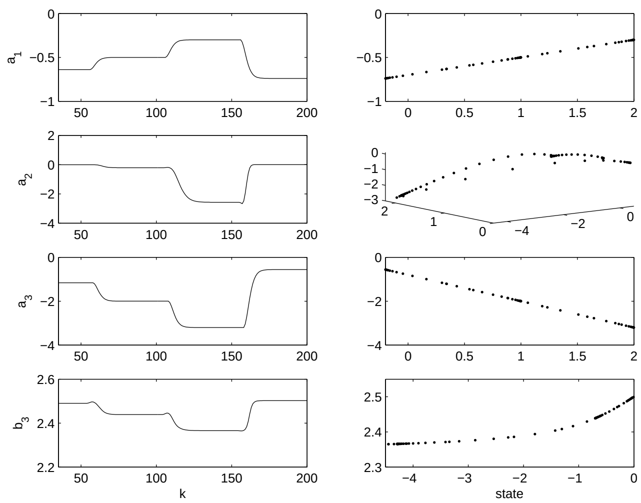

Fig. 6. State dependent parameters for the SDP/PIP control response using the slower pole positions from Fig. 5. Left hand side subplots (from top to bottom): $a_1\{\boldsymbol{\chi}_k\}$, $a_2\{\boldsymbol{\chi}_k\}$, $a_3\{\boldsymbol{\chi}_k\}$ and $b_3\{\boldsymbol{\chi}_k\}$ plotted against sample number k. Right hand side subplots: these parameters plotted against y_{k-3}, $\{y_{k-4}, u_{k-5}\}$, y_{k-5} and u_{k-4} respectively.

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