Design of an Asymptotic Sliding Mode Algorithm for Nonlinear Systems: An Observer Based Approach

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Abstract: Using the concept of dynamic sliding mode control, an asymptotic robust controller for high order nonlinear systems is presented. In this approach, a *n*-order nonlinear system is transformed into a first order system through systematic backstepping design and then it will be shown that the control objective is equivalent to design a robust control law for a 2^{nd} order auxiliary system with only output information available. To get an extra state value, a robust asymptotic observer was integrated into design process that results in an asymptotic sliding mode control algorithm. The proposed method not only preserves some features of conventional sliding mode theory but attenuates undesirable chattering action as well. A numerical example was utilized to demonstrate the applicability of the developed approach.

1. INTRODUCTION

Sliding mode control (SMC) is a particular type of variable structure control (VSC). One of the most important aspects of sliding mode is the design of fast discontinuous action, which rules controlled plant switching between two different structures, such that ideal system behaviour, called sliding mode, appears in a sliding manifold (Edwards & Spurgeon, 1998). This peculiar system characteristic leads to remarkable system performance as that in VSC. However, the discontinuous action, which is generically referred to as chattering, is also the main weakness originated by the interaction between two dynamics with imperfect switching. The discontinuous control force might degrade controlling results due to inevitable time delay existing in physical systems such that the control precision is limited. The fast switching behaviour also gives rise to a serious shortcoming in practices, such as excitation of un-modeled dynamics, wear and tear in actuator and even system instability.

For these reasons, various techniques have been thought of for chattering mitigation. Owing to its simple realization, a common skill is to set a boundary layer (Slotine & Sastry, 1983) or a smooth function around the sliding surface for generating continuous control signal, but this manner unavoidably degrades control accuracy.

Another approach to deal with the chatter is to derive a dynamic sliding mode (Sira-Ramirez, 1993). For example, the chosen sliding dynamics, usually of second order, is taken as a dynamic compensator and then the control task is to coerce the trajectories of the dynamic sliding system globally and robustly converging toward the origin of phase plane in finite time. To attain this end, estimation of extended sliding variable becomes the main control difficulty. Using a well known concept of time optimal bang-bang control for a double integrator system with only output measurable (Bartolini *et al*., 1997), a non-chattering controller is developed (Bartolini *et al*., 1998; Bartolini & Punta, 2000). But it is under the condition that an extreme value, the sign of derivative sliding variable, could be estimated with ideal precision. The control algorithm leads to a non-conventional feedback type scheme since the applied switching manifold should be renewed and memorized. Recently, an observer based SMC was proposed (Chen *et al*., 2007). However, the method cannot provide asymptotic stability of observer error since the observer design is based on a linear model. As a result, convergence of observer error was achieved by means of high observer gain. In this decade, super-twisting control algorithm (Levant, 1998; 2003) has been widely applied for design of high order SMC and state observer. Since the finite time stability proof is difficult to be illustrated by constructing Lyapunov function, the convergent behaviour is understood by geometric sense.

To preserve the simple design process of conventional SMC, avoid serious control switching, and apply for more general nonlinear systems, a synthetic control algorithm was proposed in this paper. For a given nonlinear system, an adequate nonlinear sliding surface was constructed by systematic backstepping procedures (Khalil, 2002). Then it will be shown that the stability problem of the original *n*-order nonlinear system will be reduced to a regulation problem of a 2^{nd} order auxiliary system (or a 2^{nd} order sliding dynamics). It can be taken as an extended system where the derivative of control input appears. For the auxiliary system, a SMC is applied with the aid of a robust observer such that an approximated approaching condition is achieved. The stability of closed-loop system is addressed by developing Lyapunov function and a criterion regarding selection of observer gains is also provided.

By using the proposed method, the controlled system possesses parts of advantages as in conventional SMC and removes severe chattering effect. The performance of the proposed controller is verified through a couple of numerical examples and is compared with conventional SMC.

2. SYSTEM DESCRIPTION AND CONTROL

2.1 Backstepping Design

Consider the following n-order nonlinear system with strict feedback form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3 \\
\dot{x}_3 &= f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)x_4 \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1,x_2,x_3,\ldots,x_{n-1}) + g_{n-1}(x_1,x_2,x_3,\ldots,x_{n-1})x_n \\
\dot{x}_n &= f_n(x_1,x_2,\ldots,x_n) + g_n(x_1,x_2,\ldots,x_n)u + H(t)
\end{aligned} \tag{1}$$

where $x = [x_1 \ \cdots \ x_n] \in \mathbf{R}^n$, $u \in \mathbf{R}$. The terms $f_i(0) = 0$ and $g_i(x) \neq 0$ with $i = 1,\cdots,n$, f_i, g_i are smooth functions. The unknown but smooth perturbation term $H(t)$ is differentiable up to the desired order and all its derivatives are bounded, i.e., $\sup\left|H^{(m)}(t)\right| < \mu_m$, $m = 0,1,2\cdots$, where μ_m is a bounded value and is previously known.

In the following, a nonlinear state transformation will be applied to the system (1) by utilizing backstepping approach. First, treat the system state x_2 as an independent input and then suppose that there exists a state feedback stabilizing control law $\phi_1(x_1)$ such that

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\phi_1(x_1) \tag{2}$$

is asymptotically stable.

Let a Lyapunov function be V_1 for the subsystem x_1 which satisfies $V_1 > 0$ for $x_1 \neq 0$ and

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1}\left[f_1(x_1) + g_1(x_1)\phi_1(x_1)\right] \le -Q_1(x_1) \le 0 \tag{3}$$

By adding and subtracting $g_1(x_1)\phi_1(x_1)$ (i.e., a virtual control law) to the subsystem x_1 and define a new error variable $z_1 = x_2 - \phi_1(x_1)$, the subsystem (x_1, z_1) can be represented as

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)\phi_1(x_1) + g_1(x_1)z_1 \\
\dot{z}_1 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3 - \dot{\phi}_1(x_1)
\end{aligned} \tag{4}$$

In a similar manner, consider x_3 a manipulated input and let the stabilizing control law be $\phi_2(x_1,x_2)$. Suppose that by applying the virtual control law, we are able to find a Lyapunov candidate $V_2(x_1,z_1)$ such that $V_2(x_1,z_1) > 0$ for $x_1, z_1 \neq 0$ and its corresponding time derivative satisfies

$$\begin{aligned}
\dot{V}_2 &= \frac{\partial V_2}{\partial x_1}\left[f_1(x_1) + g_1(x_1)\phi_1(x_1) + g_1(x_1)z_1\right] \\
&\quad + \frac{\partial V_2}{\partial z_1}\left[f_2(x_1,x_2) + g_2(x_1,x_2)\phi_2(x_1,z_1) - \dot{\phi}_1(x_1)\right] \\
&\le -Q_1(x_1) - Q_2(z_1) \le 0
\end{aligned} \tag{5}$$

Further define a new error variable as $z_2 = x_3 - \phi_2(x_1,z_1)$. From the control point of view, it is easily found that the asymptotic stability of the subsystem (x_1,z_1) can be guaranteed as long as the error variable z_2 is equal to zero. Note that the virtual control inputs, $\phi_1(x_1)$ and $\phi_2(x_1,z_1)$, are by no means specific form. They stand for two sets of stabilizing control law. From (4), by adding and subtracting $g_2(x_1,x_2)\phi_2(x_1,z_1)$ to the subsystem z_1, then the subsystem dynamics of (z_1,z_2) can be represented as

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)\phi_1(x_1) + g_1(x_1)z_1 \\
\dot{z}_1 &= f_2(x_1,x_2) + g_2(x_1,x_2)\phi_2(x_1,z_1) + g_2(x_1,x_2)z_2 - \dot{\phi}_1(x_1) \\
\dot{z}_2 &= f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)x_4 - \dot{\phi}_2(x_1,z_1)
\end{aligned} \tag{6}$$

Using the recursive steps until the final subsystem, one can derive the following transformed n-1 order system

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)\phi_1(x_1) + g_1(x_1)z_1 \\
\dot{z}_1 &= f_2(x_1,x_2) + g_2(x_1,x_2)\phi_2(x_1,z_1) + g_2(x_1,x_2)z_2 - \dot{\phi}_1(x_1) \\
&\vdots \\
\dot{z}_{n-2} &= f_{n-1}(x_1,\cdots,x_{n-1}) + g_{n-1}(x_1,\cdots,x_{n-1})\phi_{n-1}(x_1,\cdots,z_{n-2}) \\
&\quad + g_{n-1}(x_1,\cdots,x_{n-1})z_{n-1} - \dot{\phi}_{n-2}(x_1,\cdots,z_{n-3})
\end{aligned} \tag{7}$$

and a final subsystem in which the control input appears is

$$\begin{aligned}
\dot{z}_{n-1} &= f_n(x_1,\cdots,x_n) + g_n(x_1,\cdots,x_n)u \\
&\quad - \dot{\phi}_{n-1}(x_1,\cdots,z_{n-2}) + H(t)
\end{aligned} \tag{8}$$

where the Lyapunov function of (7) satisfies $\dot{V}_{n-1} \le -Q_1(x_1) - \sum_{j=1}^{n-2} Q_{j+1}(z_j) \le 0$ under the condition that $z_{n-1} = x_n - \phi_{n-1}(x_1,\cdots,z_{n-2}) = 0$. Therefore, by the recursive manner, the stability of the n-order system has been simplified to a regulation problem of a scalar system (8), which is the resulting transformed dynamics based on the Lyapunov stability requirements for each subsystem. In explicit words, the state $z_{n-1} = 0$ can be considered as a prescribed constraint or a sliding surface applied in conventional SMC, but it may be a nonlinear combination of state variables. Using the concept of SMC, we can find that the control objective is to keep the assigned restriction to zero in the face of external perturbations.

2.2 Sliding Mode Controller Design

Since $g_i(x) \neq 0$, according to (8), we design a control law which is in the form of

$$u = \frac{1}{g_n(x_1,\cdots,x_n)}\left(\nu_1 + \int_0^t \nu_2 d\tau\right) \tag{9}$$

where $\nu_1 = -f_n(x_1,\cdots,x_n) + \dot{\phi}_{n-1}(x_1,\cdots,z_{n-2}) - k_n z_{n-1}$ with $k_n > 0$ and ν_2 is designed to withstand exogenous disturbance. Substituting (9) into (8) yields $\dot{z}_{n-1} = -k_n z_{n-1} + \int_0^t \nu_2 d\tau + H(t)$ and its time derivative is $\ddot{z}_{n-1} = -k_n \dot{z}_{n-1} + \nu_2 + \dot{H}(t)$.

Let $[s_1 \ \ s_2] = [z_{n-1} \ \ \dot{z}_{n-1}]$, then one can derive an extended auxiliary 2nd order system as follows

$$\begin{aligned}
\dot{s}_1 &= s_2 \\
\dot{s}_2 &= -k_n s_2 + \nu_2 + \dot{H}(t)
\end{aligned} \tag{10}$$

The n-order nonlinear system has been reduced to a simple 2nd auxiliary sliding dynamics. The stability of the subsystems described in (7) is able to be guaranteed if (10) is convergent. As a result, the following objective aims to find a robust control law ν_2 for the 2nd order system by means of a conventional SMC.

With the assumption that s_2 is available and then choose an auxiliary sliding surface as

$$\sigma = s_2 + k_n s_1 \tag{11}$$

Design a control algorithm as

$$\nu_2 = -\sigma - \xi \operatorname{sgn}(\sigma) \tag{12}$$

Consider (10)-(12) and select a Lyapunov candidate as $V_\sigma = \sigma^2 / 2$, then

$$\begin{aligned} \dot{V}_\sigma &= \sigma\dot{\sigma} \\ &= \sigma\left[- k_n s_2 - \sigma - \xi \operatorname{sgn}(\sigma) + \dot{H}(t) + k_n s_2\right] \\ &\le -\sigma^2 - |\sigma|\left[\xi - \sup\left|\dot{H}(t)\right|\right] \end{aligned} \tag{13}$$

By choosing $\xi > \sup\left|\dot{H}(t)\right|$, then $\dot{V}_\sigma < 0$ is satisfied. Eq. (13) indicates that σ approaches to zero and keeps on it by t_s, where $t_s = -\ln\left[\dfrac{\xi - \sup\left|\dot{H}(t)\right|}{\xi - \sup\left|\dot{H}(t)\right| + \sqrt{|\sigma(0)|}}\right]$.

However, a main conundrum to tackle the auxiliary system is that devising such the control law requires a derivative of sliding variable, which is not directly measurable. Hence, a practical choice of control law (12) should be modified by

$$\nu_2 = -\hat{\sigma} - \xi \operatorname{sgn}(\hat{\sigma}) \tag{14}$$

where $\hat{\sigma} = \hat{s}_2 + k_n s_1$. Applying (14) into (13) and considering the condition that $|\sigma| \ge |\tilde{\sigma}|$, one can derive

$$\begin{aligned} \dot{V}_\sigma &= \sigma\left[-(\sigma - \tilde{\sigma}) - \xi \operatorname{sgn}(\sigma - \tilde{\sigma}) + \dot{H}(t)\right] \\ &\le -\sigma^2 + |\sigma||\tilde{\sigma}| - \xi|\sigma| + |\sigma| \cdot \sup\left|\dot{H}(t)\right| \\ &\le -\sigma^2 - |\sigma|\left(\xi - |\tilde{\sigma}| - \sup\left|\dot{H}(t)\right|\right) \end{aligned} \tag{15}$$

where $\tilde{\sigma} = \sigma - \hat{\sigma}$. Eq. (15) means that system trajectories approach to the error set $|\tilde{\sigma}|$ in finite time by choosing $\xi > |\tilde{\sigma}| + \left|\dot{H}(t)\right|$. It is clear that (15) equals to (13) for $|\tilde{\sigma}| = 0$.

As a result, the thickness of the estimation error must be minimized as small as possible. To attain this end, design of a robust state observer is indispensable for pursuing this purpose and is introduced in the next section.

2.3 Robust Observer Design

From the previous section, it is characterized by the fact that the discontinuous control action ν_2 affects the second time derivatives of the sliding output, and the control goal is to force s_2 converging towards zero. Nevertheless, since only s_1 is obtainable while s_2 is un-measurable through state feedback, an additional observer is required for pursuing the ideal control law (12).

Invoking (14) with (10) yields

$$\begin{aligned} \dot{s}_1 &= s_2 \\ \dot{s}_2 &= -k_n s_2 - \hat{\sigma} - \xi \operatorname{sgn}(\hat{\sigma}) + \dot{H}(t) \end{aligned} \tag{16}$$

For the closed-loop system (16), we propose an observer in the following form

$$\begin{aligned} \dot{\hat{s}}_1 &= \hat{s}_2 \\ \dot{\hat{s}}_2 &= -k_n \hat{s}_2 + k_{n+1}\tilde{s}_1 - \hat{\sigma} - \xi \operatorname{sgn}(\hat{\sigma}) + \gamma \operatorname{sgn}(\tilde{s}_1) \end{aligned} \tag{17}$$

where $\tilde{s}_1 = s_1 - \hat{s}_1$ and $\tilde{s}_2 = s_2 - \hat{s}_2$ are regarded as estimation errors of position and velocity, respectively and k_{n+1}, γ stand for observer gains. From (16) and (17), the observer error dynamics can be represented by

$$\begin{aligned} \dot{\tilde{s}}_1 &= \tilde{s}_2 \\ \dot{\tilde{s}}_2 &= -k_n \tilde{s}_2 - k_{n+1}\tilde{s}_1 - \gamma \operatorname{sgn}(\tilde{s}_1) + \dot{H}(t) \end{aligned} \tag{18}$$

The stability proof of (18) and the selection of observer gains are given in the Appendix.

Remark According to the Appendix, it has been shown that only asymptotic convergence of observer error is guaranteed, which means that $\tilde{s}_2 = 0$ can never be exactly fulfilled. As a result, the finite time approaching property (13) is lost regarding the proposed synthetic controller. However, since we have proved that σ approaches to the error set $|\tilde{\sigma}| = |\tilde{s}_2|$ in finite time, where $|\tilde{s}_2| \to 0$ as $t \to \infty$ and the convergence speed of $\tilde{s}_2$ can be made as fast as possible by increasing observer gains, the closed-loop system remains possessing similar robustness provided by conventional SMC, but the resulting control effort acts without severe chatter. Therefore, the observer based control law (9), (14) and (17) results in an asymptotic sliding mode algorithm.

3. NUMERICAL EXAMPLES

In order to illustrate the advantage of the proposed asymptotic controller, a few examples are considered. One is control of a highly nonlinear system with the consideration of conventional SMC design and the others are with the same system dynamics for regulation and tracking tasks but utilizes the proposed observer based sliding mode algorithm.

In following examples, we focus on external disturbance rejection such that the case $f_3 = 0$ and $g_3 = 1$, i.e., $n = 3$, is considered for the rest of simulations.

Example 1: Consider the following 3rd order nonlinear system subjected to external disturbance

$$\begin{aligned} \dot{x}_1 &= x_1^3 + x_2 \\ \dot{x}_2 &= -x_1^4 + x_2^2 x_3 + x_3 \\ \dot{x}_3 &= u + \delta \end{aligned} \tag{19}$$

Referring to (1), it is easy to find that $f_1 = x_1^3$, $g_1 = 1$, $f_2 = -x_1^4$, $g_2 = 1 + x_2^2$ and $H(t) = \delta(t) = 20\cos(\omega_d t)$ with $\omega_d = 4$ rad/s. System initial condition was set to be $\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) \end{bmatrix} = \begin{bmatrix} 0.25 & -0.5 & 0 \end{bmatrix}$.

Following the steps (2)-(8), we have

$$\begin{aligned} \dot{x}_1 &= x_1^3 + z_1 + \phi_1(x_1) \\ \dot{z}_1 &= -x_1^4 + \left(x_2^2 + 1\right)\left(z_2 + \phi_2(x_1, z_1)\right) - \dot{\phi}_1(x_1) \end{aligned} \tag{20}$$

and the last subsystem is

$$\dot{z}_2 = u + \delta - \dot{\phi}_2(x_1, z_1) \tag{21}$$

where the variables used in (20) and (21) are defined by $z_1 = x_2 - \phi_1(x_1)$, $z_2 = x_3 - \phi_2(x_1, z_1)$, $\phi_1(x_1) = -x_1^3 - k_1 x_1$, and $\phi_2(x_1, z_1) = (x_2^2 + 1)^{-1}(x_1^4 - x_1 + \dot{\phi}_1(x_1) - k_2 z_1)$. By the previous analysis, we have known that the main task is to find a proper control law such that $z_2 \to 0$.

Using the concept of SMC, we design

$$u = \dot{\phi}_2(x_1, z_1) - k_3 z_2 - w\,\mathrm{sgn}(z_2) \tag{22}$$

where $k_1 = 3$, $k_2 = 5$ and $k_3 = 250$ are applied.

Select a Lyapunov as

$$V_z = z_2^2 / 2 \tag{23}$$

From (21)-(23), the approaching condition

$$\begin{aligned} \dot{V}_z &= z_2\left(u + H - \dot{\phi}_2(x_1, z_1)\right) \\ &\le -k_3 z_2^2 - |z_2|\left(w - \sup|H|\right) \\ &\le 0 \end{aligned} \tag{24}$$

is guaranteed with $w = 25 > \sup|H|$ and thus z_2 reaches zero in finite time.

Example 2: By the conventional sliding mode theory, *Example 1* has shown that the control law (22) guarantees the finite time approaching condition. However, it apparently acts with chatter. Therefore, we replace (22) by the proposed control configuration (9) and (14) as follows

$$\begin{aligned} u &= v_1 + \int_o^t v_2 d\tau \\ &= \dot{\phi}_2(x_1, z_1) - k_3 z_2 - \int_o^t (\hat{\sigma} + \xi\,\mathrm{sgn}(\hat{\sigma}))d\tau \end{aligned} \tag{25}$$

and the corresponding observer is

$$\begin{aligned} \dot{\hat{s}}_1 &= \hat{s}_2 \\ \dot{\hat{s}}_2 &= -k_3 \hat{s}_2 + k_4 \tilde{s}_1 - \hat{\sigma} - \xi\,\mathrm{sgn}(\hat{\sigma}) + \gamma\,\mathrm{sgn}(\tilde{s}_1) \end{aligned} \tag{26}$$

where $k_4 = 15000$ and $\hat{\sigma} = \hat{s}_2 + k_3 s_1$. Referring (A.2), there exists $\beta = \left(k_3 + \sqrt{k_3^2 - 4k_4}\right)/2 = 150$. According to (15), ξ should be chosen properly such that $\xi > |\tilde{\sigma}| + \sup|\dot{H}(t)|$ is met. Actually, since the amplitude of $|\tilde{\sigma}|$ decreases rapidly, we have $|\tilde{\sigma}| + \sup|\dot{H}(t)| \approx \varepsilon + \sup|\dot{H}(t)|$ after transient response, where ε is a small positive value. Thus, $\xi = 85$ was used. On the other hand, based on the criterion (A.4), we selected $\gamma = 85 > \sup\left(|\dot{H}(t)| + \beta^{-1}|d\dot{H}(t)/dt|\right)$. Therefore, we can guarantee that $\tilde{s}_2$ and $\tilde{s}_1 \to 0$ asymptotically. Fig. 1 and 2 separately show the estimation responses of s_1 and s_2, and it is obviously that $\hat{s}_1$ and $\hat{s}_2$ converge to s_1 and s_2 closely after 0.05sec. Therefore, the approximated approaching phenomenon takes place after 0.05sec as shown in Fig. 3. For $z_2 = 0$, system (20) can be rewritten by

$$\begin{aligned} \dot{x}_1 &= x_1^3 + z_1 + \phi_1(x_1) \\ \dot{z}_1 &= -x_1 - k_2 z_1 \end{aligned} \tag{27}$$

By choosing $V_{x_1,z_1} = x_1^2 + z_1^2$, it is easy to find that $\dot{V}_{x_1,z_1} = x_1\dot{x}_1 + z_1\dot{z}_1 = -k_1 x_1^2 - k_2 z_1^2 \le 0$. Therefore, x_1, $z_1 \to 0$ which also indicates that $x_2 = \phi_1(x_1) \to 0$ and $x_3 = \phi_2(x_1, z_1) \to 0$. Fig. 4 depicts the closed-loop states response and the applied control force is illustrated in Fig. 5. Comparing (25) with (22), it is easily found that the proposed control law not only successfully forces system states towards zero asymptotically in spite of the presence of unknown disturbance but removes chattering action as well.

Example 3: For output tracking task, let $e = y - x_d = x_1 - x_d$, where x_d stands for smooth desired trajectory and its time derivatives are previously known. Then we can derive the similar dynamics

$$\begin{aligned} \dot{e} &= x_1^3 + z_1 + \phi_1(x_1) - \dot{x}_d \\ \dot{z}_1 &= -x_1^4 + (x_2^2 + 1)(z_2 + \phi_2(x_1, z_1)) - \dot{\phi}_1(x_1) \\ \dot{z}_2 &= u + \delta - \dot{\phi}_2(x_1, z_1) \end{aligned} \tag{28}$$

where the definition of state variables are the same with that used in the previous examples, except the stabilizing control terms, which have been modified by

$$\phi_1(x_1, x_d) = -x_1^3 - k_1 e + \dot{x}_d$$

$$\phi_2(x_1, z_1, x_d) = (x_2^2 + 1)^{-1}(x_1^4 - e + \dot{\phi}_1(x_1) - k_2 z_1)$$

in this case. In the simulations, $x_d = A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t)$ with $A_1 = 0.1$, $A_2 = 0.2$, $\omega_1 = 1$ (rad/s), $\omega_2 = 0.35$ (rad/s) and system initial condition $[x_1(0) \;\; x_2(0) \;\; x_3(0)] = [0.75 \;\; -1.5 \;\; 0]$ were applied. Fig. 6 shows that the controlled system output successfully tracks the given trajectory. Regarding internal dynamics, since the relative degree between the defined output and control input is three, no unstable behaviour will be induced during control process and thereby the resulting control signal is also bounded. Fig. 7 depicts the stable response of internal states. Note that if the system output is replaced by $y = x_2$, the corresponding relative degree becomes two, which is smaller than system order. For such the situation, it is easy to find that the internal dynamics is unstable since the corresponding zero dynamics is unstable, i.e., $\dot{x}_1 = x_1^3$.

4. CONCLUSION

In the conventional SMC, several robust characteristics are guaranteed if the chosen constraint can be kept zero in the presence of uncertain dynamics and unknown exogenous disturbances. Inheriting this simple idea, in this paper, the constraint for a nonlinear system is constructed through systematic backstepping design such that the asymptotical stability of nonlinear systems can be guaranteed if the prescribed constraint steers to zero. Regarding the proposed synthetic control algorithm, it can only provide asymptotic stability for sliding dynamics since the observed error converges towards origin asymptotically. Thus, the finite time property is lost theoretically. Fortunately, the convergence speed can be increased as fast as possible by adjusting observer gains such that the closed-loop dynamics acts similar to the system controlled by conventional SMC. However, different from the SMC, the developed method is capable of avoiding serious control chattering.

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Appendix. STABILITY PROOF OF OBSERVER

For (18), we define a new state as

$$\eta = \tilde{s}_2 + \beta\,\tilde{s}_1 \tag{A.1}$$

where $\beta > 0$ and it satisfies

$$\beta^2 - k_n\beta + k_{n+1} = 0 \tag{A.2}$$

From (18) and (A.2), the time derivative of (A.1) is

$$\begin{aligned}\dot{\eta} &= -k_n\tilde{s}_2 - k_{n+1}\tilde{s}_1 - \gamma\,\mathrm{sgn}(\tilde{s}_1) + \dot{H}(t) + \beta\,\dot{\tilde{s}}_1 \\ &= -(k_n - \beta)\tilde{s}_2 - k_{n+1}\tilde{s}_1 - \gamma\,\mathrm{sgn}(\tilde{s}_1) + \dot{H}(t) \\ &= -(k_n - \beta)\eta - \gamma\,\mathrm{sgn}(\tilde{s}_1) + \dot{H}(t)\end{aligned} \tag{A.3}$$

For further address stability analysis of (A.3), the following theorem is invoked.

Theorem In a similar manner presented in (Su *et al.*, 2007), we are going to show that the observer (18) ensures global asymptotic convergence of $\tilde{s}_2(t)$ in the sense that $\tilde{s}_1(t)$, $\tilde{s}_2(t) \to 0$ as $t \to \infty$, provided the prescribed gain γ satisfied the following sufficient condition:

$$\gamma > \sup\left(\left|\dot{H}(t)\right| + \beta^{-1}\left|d\dot{H}(t)/dt\right|\right) \tag{A.4}$$

Proof. Let the function $L(t) \in R$ be defined as follows

$$L := \eta\left(\dot{H}(t) - \gamma\,\mathrm{sgn}(\tilde{s}_1(\tau))\right) \tag{A.5}$$

Then the time integration of (A.5) satisfies

$$\begin{aligned}\int_0^t L(\tau)d\tau \le{}& \left(\left|\dot{H}(t)\right| - \gamma\right)\left|\tilde{s}_1(t)\right| + \gamma\left|\tilde{s}_1(0)\right| - \dot{H}(0)\tilde{s}_1(0) \\ &+ \beta\int_0^t \left|\tilde{s}_1(\tau)\right|\left(\left|\dot{H}(\tau)\right| + \left|\beta^{-1}\frac{d\dot{H}(\tau)}{d\tau}\right| - \gamma\right)d\tau\end{aligned} \tag{A.6}$$

Taking (A.4) into consideration, we further get

$$\int_0^t L(\tau)d\tau \le \gamma\left|\tilde{s}_1(0)\right| - \dot{H}(0)\tilde{s}_1(0) \tag{A.7}$$

Eq. (A.7) implies that providing (A.4) is satisfied, then there must exist a bounded positive constant $\delta := \gamma\left|\tilde{s}_1(0)\right| - \dot{H}(0)\tilde{s}_1(0)$ such that

$$\varepsilon := \delta - \int_0^t L(\tau)d\tau \ge 0 \tag{A.8}$$

By choosing a Lyapunov candidate as

$$V_o = \eta^2/2 + \varepsilon \tag{A.9}$$

According to (A.3) and (A.8), the time derivative of (A.9) is

$$\begin{aligned}\dot{V}_o &= \eta\left(-(k_n - \beta)\eta - \gamma\,\mathrm{sgn}(\tilde{s}_1) + \dot{H}(t)\right) - L(\tau) \\ &= -(k_n - \beta)\eta^2\end{aligned} \tag{A.10}$$

It is clear that the convergence speed is associated with the selection of k_n and β. From (A.2) and (A.10), it is true that there exists $\beta = \left(k_n + \sqrt{k_n^2 - 4k_{n+1}}\right)/2 < k_n$ such that $\eta \to 0$. From (A.1), provided η steers almost to zero after $t = t_{\eta\to 0}$, we can deduce that $\tilde{s}_1(t - t_{\eta\to 0}) = \tilde{s}_1(t_{\eta\to 0})e^{-\beta(t - t_{\eta\to 0})} \to 0$ and $\tilde{s}_2(t - t_{\eta\to 0}) = -\beta\tilde{s}_1(t_{\eta\to 0})e^{-\beta(t - t_{\eta\to 0})} \to 0$. Therefore, the error bound $|\tilde{\sigma}| = |\tilde{s}_2| \to 0$. Note that for a properly given larger pair of (k_n, k_{n+1}), a larger value of β is generated, which further indicates that the required amplitude of γ can be attenuated as shown in (A.4).

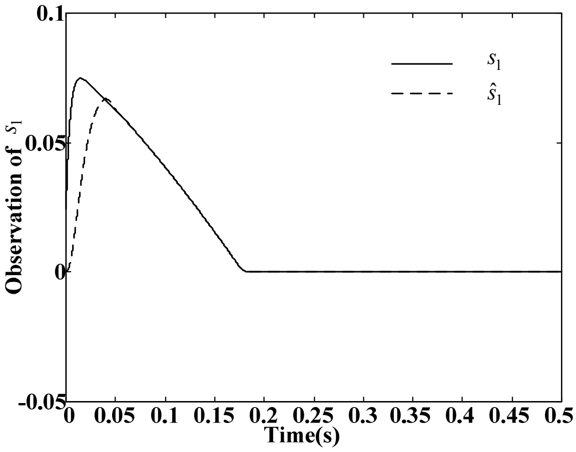

Fig. 1. Convergent behaviour of $\hat{s}_1$ with $\xi = 85$ is applied.

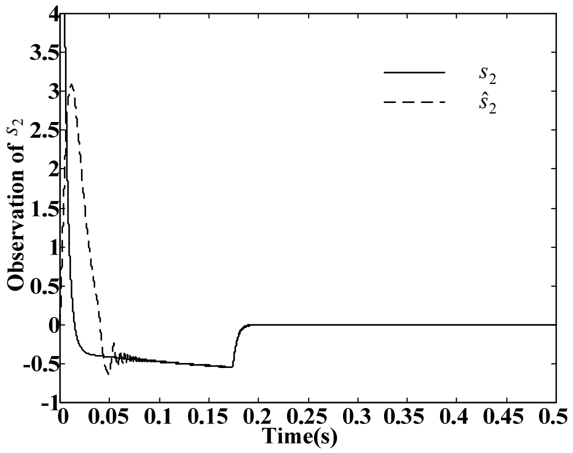

Fig. 2. Convergent behaviour of $\hat{s}_2$ with $\xi = 85$ is applied.

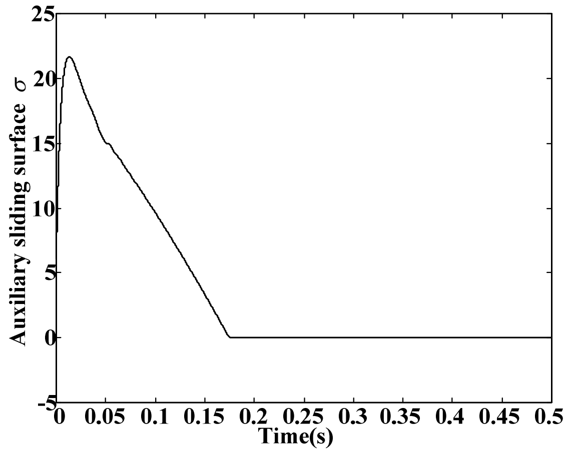

Fig. 3. Response of sliding variable.

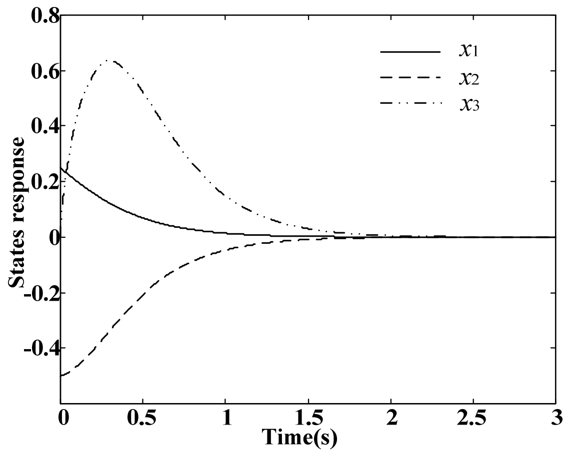

Fig. 4. Convergent behaviour of the controlled nonlinear system.

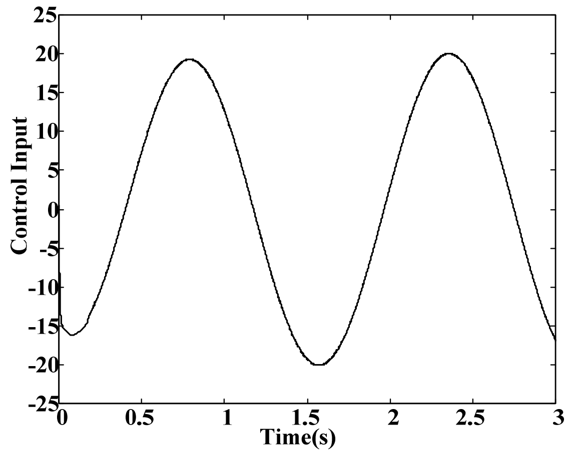

Fig. 5. Proposed synthetic control law.

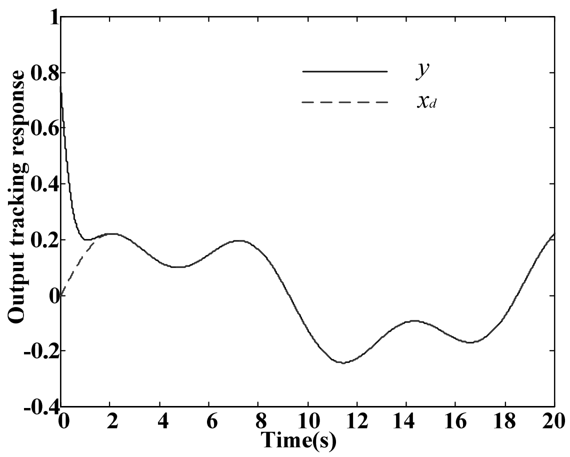

Fig. 6. Tracking response.

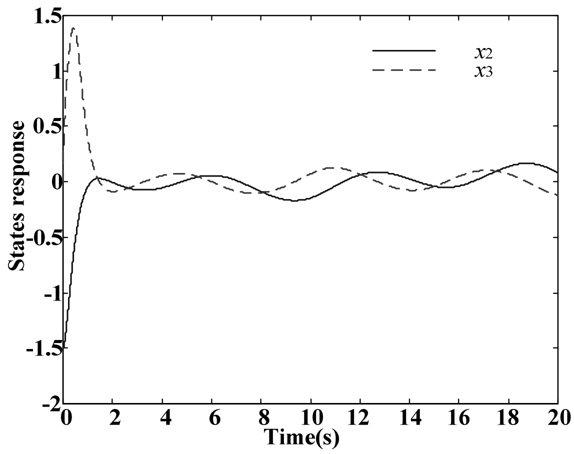

Fig. 7. Responses of x_2 and x_3.