ABOUT EQUIVALENCE BETWEEN SLIDING MODE AND CONTINUOUS CONTROL SYSTEMS

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Abstract: It is shown, how to create the continuous system equivalent to the system with sliding mode control. In the case of minimum phase plants, the system arises from the replacement of the relay with small hysteresis by the amplifier with high gain, connected in series with saturation having appropriate parameters. In the case of nonminimum phase (or other difficult plants) it is noted that similar equivalence exists for the continuous and relay system with parallel compensator. The latter system may be treated as the system with modified sliding mode control. In the equivalent continuous systems the chattering effect, related with sliding mode control doesn't exist.

Keywords: Sliding mode, continuous control, linear, nonlinear, control saturation.

1. INTRODUCTION

The control systems which use sliding mode technique have now good theoretical elaborations (Slotine and Lee, 1991; Utkin, 1992), as well as successful practical applications (e.g. commonly used voltage regulation of car alternators). This kind of systems operates well both with linear and nonlinear plants.

It is well known that the systems with sliding mode control are very robust so they operate well even in the case of large and rapid parameter changes. However with the switching action of the relay, there is connected the so called chattering effect (Åström, 1989), which sometimes is not accepted by users and/or actuators. Therefore chattering decrease is interesting from application point of view.

There was many trials of decreasing of chattering effect for instance by applying different amplitudes of the relay output for different absolute values of the error (Åström, 1989). Another approach is based on introduction of so called boundary layers in which the dependence between converted error and control is continuous (Slotine and Lee, 1991). It was noted that the obtained in this manner control approximates the sliding mode control.

The considerations of the present paper are related with the latter approach. In the case of minimum phase plants for which the sliding mode control may be applied, the continuous system with appropriate parameters is created , for which both the controls: sliding mode and continuous have the same outputs under the same excitations.

For the case of nonminimum phase plants, when the usual sliding mode control cannot be applied the parallel compensator (Gessing, 2007) has been used to implement modified sliding mode control (relay control with fast switching without the need of higher order derivatives approximation used in usual sliding mode control). It is also shown that in this case there is similar equivalence between continuous and relay control systems.

In the described equivalent continuous control systems the chattering disappears, though from the point of view of the plant outputs they have the same properties as the system with sliding mode control. This is advantage of the considered continuous control. In connection with that there arises the question whether the application of the systems with sliding mode control may be justified at all.

2. SLIDING MODE CONTROL

The block diagram of the system with sliding mode control and the characteristic of the relay are shown in Fig. 1 a and b, respectively. Here the signals u, y, r, $e = r - y$ are the input, output of the plant, reference value and error, respectively. Roughly speaking, the sliding mode control is based on fast switching of the relay, so that the generated fast frequency harmonics appearing in the signal u are filtered by the dynamics of the plant G and the output y depends mainly on the averaged value of the input u. The fast switching is obtained owing to the choice of polynomial $C(s)$, for which the change of the initial slope of the response of the signal e^* to the stepwise change of u is nonzero and owing to the appropriate choice of the switched magnitudes $-H$ and $+H$ of the relay. The choice of $C(s)$ will be described in details further on.

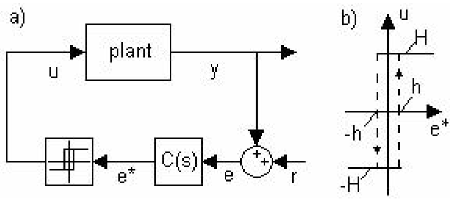

Fig. 1. a) The system with sliding mode control; b) characteristic of the relay.

2.1 The Case of Linear Plant

Consider the linear plant described by the transfer function (TF)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{L(s)}{M(s)} \quad (1)$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the plant output $y(t)$ and input $u(t)$, respectively, while $L(s)$ and $M(s)$ are polynomials of m-th and n-th degree, respectively, $m < n$, $d = n - m$ is the relative degree of the TF $G(s)$. Assume that the TF $G(s)$ has minimum phase zeros.

Block $C(s)$ from Fig. 1a is determined by the following polynomial

$$C(s) = c_0 s^{d-1} + c_1 s^{d-2} + ... + c_{d-2} s + 1 \quad (2)$$

so that the corresponding equation

$$e^* = c_0 e^{(d-1)} + c_1 e^{(d-2)} + ... + c^{d-2} e^{(1)} + e \quad (3)$$

for $e^* = 0$ has stable transients of a good quality (i.e. with sufficiently fast decay). It may be for instance $C(s) = (Ts + 1)^{d-1}$, with multiple root $s_1 = -1/T$, where T is possibly small time constant, which gives fast decay of the transient.

Note that, under slowly varying r (and in an appropriate region), for the hysteresis of the relay $h \to 0$ it appears fast switching and we have $e^* \to 0$, i.e. $e \to 0$ and $y \to r$. Since the relay then operates on the vertical segments of its characteristic, then it may be replaced by the linear amplifier with high gain k ($k \to \infty$).The stability of the resulting linear system may be easy analyzed. Really the characteristic equation of the closed loop (CL) system takes the form

$$M(s) + kL(s)C(s) = 0 \quad (4)$$

Note, that if $k \to \infty$ then m roots of (4) tend to zeros of $L(s)$, while $(d - 1)$ roots of (4) tend to zeros of $C(s)$. This means that the resulting linear system for high gain k (and the analyzed system with sliding mode control) may be stable only then when the plant TF $G(s)$ has minimum phase zeros. This observation justifies the assumption about minimum phase zeros of the plant $G(s)$, needed for sliding mode control. Moreover, if the open loop TF $G(s)C(s)$ has relative degree equal to one, then the obtained CL system with minimum phase plant may be stable even for very high gain k (Gessing, 2006).

In implementation, the higher order derivatives appearing in (3) may be approximated by substituting $s/(1+s\tau)$ in (2), in the place of the operator s. Here τ denotes a very small time constant.

Note, that during fast switching the transients of the system with sliding mode control are described by the differential equation (2) (for $e^* = 0$) with parameters c_i, $i = 1, 2, ..., d - 2$ independent of the parameters of the plant. Therefore the system is very robust. By the appropriate choice of the parameters c_i we may obtain very good transients.

2.2 The Case of Nonlinear Plant

Now, consider the nonlinear plant G described by the following state equations

$$\dot{x} = f(x, u), \quad y = g(x) \quad (5)$$

where x is n-dimensional state and u, y are scalar input and output, while $f(x, u)$, $g(x)$ are vector and scalar functions of the mentioned arguments, respectively. Also now, the sliding mode control may be implemented in the system shown in Fig. 1, with the same formulas (2), (3), describing the polynomial $C(s)$, signal e^* and the same described above relay.

One difference in comparison to system with linear plant described in previous subsection is determination of the relative degree d – the notion which is also used in the case of nonlinear plant (Slotine and Lee, 1991). As in the case of linear plant d is an integer such that it appears a stepwise change of the d-th derivative $y^{(d)}$, when it appears a stepwise change of the input u. The integer d may be obtained from successive differentiation of the output y with accounting (5).

Note, that most of the remarks formulated in previous subsection and concerned implementation of the sliding mode control are also valid for the system with nonlinear plants.

The system shown in Fig. 1a, with linear or nonlinear plant will be called the relay system with sliding mode control. In this system, when $h \to 0$ the frequency of oscillations tends to infinity and the oscillations resulting from this switching and appearing in the input signal are filtered by the dynamics of the plant. Denote by $\bar{y}(t)$ the output of the plant (linear or nonlinear) with filtered oscillations.

3. THE EQUIVALENT CONTINUOUS SYSTEM

Now, consider the system shown in Fig. 2, which will be called the continuous system with higher order derivatives in regulator and saturation.The block k denotes the amplifier with high gain k. To constrain the control u the additional block containing control saturation is introduced. It is described by the formulas: $u = u_{mx}$ for $v \geq u_{mx}$, $u = v$ for $u_{min} \leq v \leq u_{mx}$ and $u = u_{min}$ for $v \leq u_{min}$. The plant may be linear with TF (1), or nonlinear with state equations (5). Good properties of this kind of systems were discussed in (Gessing, 2006).

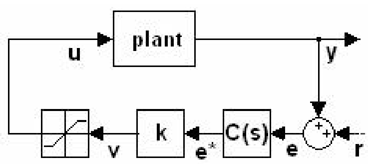

Fig. 2. Continuous system with higher order derivatives in regulator and saturation.

Note, that if we choose $k = H/h$, $u_{min} = -H$, $u_{mx} = H$, then two blocks corresponding to gain k and saturation in Fig. 2 are described by the formulas: $u = -H$ for $e^* \leq -h$, $u = ke^*$ for $|e^*| \leq h$ and $u = H$ for $e^* \geq h$. Comparing with description of the relay and accounting that in the region $|e^*| \leq h$ the sliding mode causes switching and linearization, we obtain:

Corollary 1. If in the relay system $h \to 0$ and in the continuous system $u_{mx} = +H$, $u_{min} = -H$ and $k \to \infty$ then from the point of view of output waveforms, the relay system with sliding mode control is equivalent to the continuous system with higher order derivatives in regulator and saturation. This means that for the same external excitations (reference values or disturbances), for both the systems the plant output $\bar{y}(t)$ of the relay system tends to the output $y(t)$ of the continuous system.

Of course this means that for small hysteresis h, high gain k and $u_{mx} = H^+$, $u_{min} = H^-$, the outputs $\bar{y}(t)$ is very close to $y(t)$. At the same time one may note that the control signals $u(t)$ in both the systems are completely different. The input $u(t)$ in relay system has high frequency switching, which however after filtering high frequency harmonics tends also to the input of the continuous system. In the periods where $u = u_{mx}$, or $u = u_{min}$ the inputs of both the systems take the same values. This will be confirmed in the following example.

4. EXAMPLE 1

Consider the nonlinear plant of second order described by the state equations

$$\dot{x}_1 = -2x_1 + 4(1 + |u|)u,$$
$$\dot{x}_2 = (x_1 - x_2)\frac{1}{1 + |x_2|}, \quad y = x_2 \qquad (6)$$

The plant model may be interpreted as in series connection of two first order lag elements – first of them has the "gain" $4(1+|u|)$ dependent on u and second the "time constant" $(1 + |x_2|)$ dependent on x_2. One may note that for stepwise change of u there appear stepwise change of $\ddot{y}$ (while $\dot{y}$ is continuous). This means that $d = 2$ and we choose

$$C(s) = Ts + 1, \qquad T = 0.25 \qquad (7)$$

though, one can check that also now, there is great freedom in choosing polynomial $C(s)$ (smaller T, faster response).

In simulations performed in SIMULINK the polynomial $C(s)$ (7) was approximated using formula $s \approx s/(1 + s\tau)$, $\tau = 0.01$. In simulations the two systems: relay with sliding mode and continuous with derivative in regulator and saturation were compared. For both the systems the experiments were performed for the following data: $r(t) = 12 \cdot \mathbf{1}(t - 1)$, $(\mathbf{1}(t) = 0$ for $t < 0$ and $\mathbf{1}(t) = 1$ for $t > 0)$, $u_{mx} = +H = 5$, $u_{min} = -H = -5$, $h = 0.05$, $k = 100$.

In Fig. 3a the time responses for both the systems relay and continuous are compared. It is shown that the responses of both the systems relay and continuous are very close one to other. From Fig. 3b it results that for both the systems relay and continuous the controls are the same in the time intervals where u=5, or u=-5

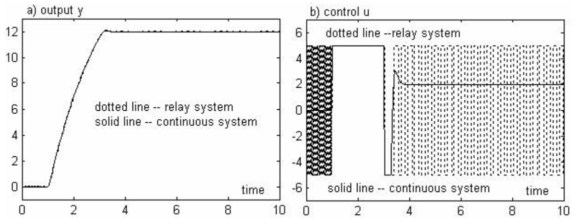

Fig. 3. Comparison of a) outputs y and b) controls u for relay and continuous systems from Example 1.

5. CONTINUOUS SYSTEM WITH NONMINIMUM PHASE PLANT

To control nonminimum phase or other difficult plants, the parallel compensator has been introduced in (Gessing, 2007). This idea will be reminded here for a continuous system and used to create a system with modified sliding mode control which will be equivalent to the continuous one.

The continuous system with parallel compensator and control saturation is shown in Fig. 4a and b.

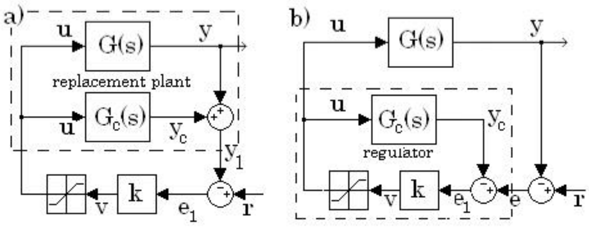

Fig. 4. The equivalent block diagrams of the continuous systems with parallel compensator and saturation and with outlining a) the replacement plant; b) the resulting regulator.

We limit our considerations to linear, stable, but difficult plants (i.e. nonminimum phase and/or with delay and/or of higher order) described by the TF (1). The parallel compensator is described by the following TF

$$G_c(s) = \frac{Y_c(s)}{U(s)} = G_1(s) - G(s) \tag{8}$$

Here $Y_c(s)$ is the Laplace transform of the output y_c of the compensator, while $G_1(s)$ is the TF which will be appropriately chosen. Note that in the proposed structure shown in Fig. 4a the TF of the replacement plant outlined by the dashed line is described by

$$\frac{Y_1(s)}{U(s)} = G(s) + G_c(s) = G_1(s) \tag{9}$$

In the case of regulation, when the reference signal $r = const$ the TF $G_1(s)$ should fulfill the following condition

$$G_1(0) = G(0) \tag{10}$$

so that for steady state values it is

$$y_c = 0, \quad y_1 = y, \quad e_1 = r - y_1 = r - y \tag{11}$$

Since in the considered system a high gain will be used the replacement plant TF $G_1(s)$ should be chosen appropriately to obtain stable CL system with proper phase margin. This will be fulfilled if the TF $G_1(s)$ has the relative degree equal to one and its parameters are appropriately chosen, as described further on.

5.1 Approximate Description of the Closed Loop System

The equivalent block diagram of the system from Fig. 4a is shown in Fig. 4b. Note that the part of the system outlined by the dashed line contains the elements of the regulator based on the parallel compensator. Assuming that the system has appropriate phase margin, under high gain k, and that the system operates in the linear region (i.e. neglecting the element with saturation) the regulator in the system is described by the following TF

$$G_r(s) = \frac{U(s)}{E(s)} = \frac{k}{1 + kG_c(s)} \approx \frac{1}{G_c(s)} \tag{12}$$

Accounting (12) we obtain the following formula describing the CL system

$$\frac{Y(s)}{R(s)} = \frac{G(s)/G_c(s)}{1 + G(s)/G_c(s)} = \frac{G(s)}{G_1(s)} \tag{13}$$

where $R(s) = \mathcal{L}[r(t)]$ and $\mathcal{L}$ denotes Laplace transform.

5.2 Design of the Replacement Plant Transfer Function

Denote by
$$G_1(s) = \frac{L_1(s)}{M_1(s)} \tag{14}$$

a stable replacement plant (9) with minimum phase zeros. One way of designing $G_1(s)$ is to choose

$$M_1(s) = M(s) \tag{15}$$

$$L_1(s) = l(1 + sT)^{n-1}, \quad l = L(0) \tag{16}$$

so the condition (10) is fulfilled.

Denote by $\varphi_1(\omega)$ the phase of the frequency response $G_1(j\omega) = L_1(j\omega)/M(j\omega)$. Let the phase $\varphi_1(\omega)$ fulfills the inequality

$$-180^\circ < \varphi_1(\omega) \le 0 \tag{17}$$

Since $G_1(s)$ has the relative degree equal to one and $lim_{\omega\to\infty}\varphi_1(\omega) = -90^\circ$ then the CL system shown in Fig. 4a (and 4b) is stable and may have demanded phase margin for very high gain k. This results from Nyquist criterion of stability and from the shape of the frequency response of $G_1(j\omega)$.

Accounting (1), (14), (15) in (13) we obtain for the CL system

$$\frac{Y(s)}{R(s)} = \frac{L(s)}{L_1(s)} \tag{18}$$

From these considerations it results that in the considered case the choice of $L_1(s)$ influences the dynamics of the considered CL system, essentially. Really the characteristic equation of the CL system is

$$L_1(s) = 0 \tag{19}$$

and its roots influence the velocity of decay of the transient response. Therefore we choose $L_1(s)$ in the form (16) containing the multiple root $s_1 = -1/T$. Of course, to obtain fast transient, we should choose a possibly small time constant T, for which the condition (17) is fulfilled. For the chosen T the condition (17) may be easy checked using MATLAB command `nichols(.)` (or `nyquist(.)`). Some modifications in choosing $L_1(s)$ are possible (Gessing, 2007). It becomes that appearance of saturation does not change the property of the system, essentially.

6. MODIFIED SLIDING MODE CONTROL FOR NONMINIMUM PHASE PLANT

Now, consider the system shown in Fig. 5 in which it appears the relay in the place of amplifier and saturation of the system shown in Fig. 4b. One may think that this system implements a modified sliding mode control for nonmimnimum phase plant. As the replacement plant TF $G_1(s)$ has the relative degree equal to one, high frequency oscillations are generated by fast switching of the relay, for sufficiently small hysteresis h. They are filtered by the dynamics of the plant $G(s)$ and compensator $G_c(s)$. Let $\bar{y}(t)$ and $\bar{y}_c(t)$ be the outputs of the plant $G(s)$ and parallel compensator $G_c(s)$, respectively, in which the high frequency oscillations are neglected. Since the amplitudes of the oscillations are small then it is

$$\bar{y}(t) \approx y(t), \quad \bar{y}_c(t) \approx y_c(t) \tag{20}$$

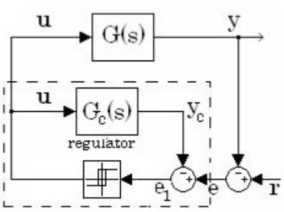

Fig. 5. a) Relay system with modified sliding mode control

During fast switching the relay operates on vertical segments of its characteristic, therefore in an approximate description the relay may be treated as the linear amplifier with very high gain k ($k \to \infty$ when $h \to 0$).

Let $\bar{u}(t)$ be the control signal with filtered high frequency oscillations, containing the slowly varying component such that $\bar{Y}(s) = G(s)\bar{U}(s)$, where $\bar{Y}(s) = \mathcal{L}[\bar{y}(t)]$ and $\bar{U}(s) = \mathcal{L}[\bar{u}(t)]$ Let $\bar{Y}_c(s) = \mathcal{L}[\bar{y}_c(t)] = G_c(s)\bar{U}(s)$ and $\bar{E}(s) = R(s) - \bar{Y}(s), R(s) = \mathcal{L}[r(t)]$, $\bar{E}_1(s) = \bar{E}(s) - \bar{Y}_c(s)$. During fast switching it is $|e_1| \le h$ and if $h \to 0$ we have $e_1 \approx 0$, $E_1(s) \approx 0$ and $\bar{E}_1(s) \approx 0$. Since $\bar{E}_1(s) = \bar{E}(s) - G_c(s)\bar{U}(s) \approx 0$ then

$$C(s) = \frac{\bar{U}(s)}{\bar{E}(s)} \approx \frac{1}{G_c(s)} \tag{21}$$

The formula (21) describes the TF of the regulator outlined in Fig. 5 by the dashed line. The TF (21) takes the same form as the formula (12) valid for continuous system shown in Fig. 4b. Therefore for the CL system we have

$$\frac{\bar{Y}(s)}{R(s)} = \frac{G(s)/G_c(s)}{1 + G(s)/G_c(s)} = \frac{G(s)}{G_1(s)} \tag{22}$$

Then the variables $\bar{u}(t), \bar{e}(t), \bar{y}(t)$ and r(t), in which the high frequency oscillations are filtered, are related by the same TF-s as in the linear continuous system from Fig 4b the variables $u(t), e(t), y(t)$ and $r(t)$ are (compare with formulas (12), (13)). Thus the parallel compensator for relay system may be designed in the same manner as previously (for the continuous system).

Strictly speaking the formulas (21), (22) are valid for the system shown in Fig. 5, if the relay generates fast switching of the control u from $u = -H$ to $u = +H$. In this case the relay may be replaced by the linear amplifier with high gain k. However from the characteristic of the relay it results that the maximal and minimal values of the control u are determined by $+H$ and $-H$, respectively. In the system shown in Fig. 5 the relay sometimes gives the values of the control equal to $+H$ or $-H$, in some period of time without fast switching. One may note that both the cases with and without fast switching may be accounted by replacing the relay in the system shown in Fig. 6 by the amplifier with high gain k (if $h \to 0$ then $k \to \infty$) connected in series with the element with saturation $u_{mx} = H$ and $u_{min} = -H$, as in the continuous system shown in Fig. 4b.

Corollary 2. If in the relay system shown in Fig. 5 the hysteresis of the relay $h \to 0$ and in the continuous system shown in Fig. 4b $u_{mx} = +H$, $u_{min} = -H$ and $k \to \infty$, then from the point of view of the outputs y, the relay system (Fig. 5) is equivalent to the continuous system (Fig. 4b). This means, that for the same external excitations (reference values or disturbances) appearing in both the control systems, the plant output $\bar{y}(t)$ of the relay system tends to the plant output $y(t)$ of the continuous system.

7. EXAMPLE 2

Consider the plant described by the following TF

$$G(s) = \frac{-0.5s + 1}{s^4 + 3.5s^3 + 6s^2 + 7s + 3} \tag{23}$$

The TF $G(s)$ has the following stable poles $p_1 = -1.8185$, $p_2 = -0.7343$, $p_3 = -0.4736 + j1.4221$, $p_4 = -0.4736 - j1.4221$ and the one nonminimum phase zero $z_1 = 2$. The plant is of forth order and nonminimum phase.

To design the parallel compensator we choose the TF $G_1(s)$ in the form determined by (15), (16). Time constant T has been chosen after several trials with using MATLAB `nichols(.)` command. From these trials it results that for $T = 0.3$ the minimal phase of $G_1(j\omega)$ is equal to $-167°$ and the condition (17) is fulfilled with some margin. Accounting (16) with $T = 0.3$ we obtain

$$G_1(s) = \frac{0.027s^3 + 0.27s^2 + 0.9s + 1}{s^4 + 3.5s^3 + 6s^2 + 7s + 3} \tag{24}$$

Then the formula (8) together with (23) and (24) gives

$$G_c(s) = \frac{0.027s^3 + 0.27s^2 + 1.4s}{s^4 + 3.5s^3 + 6s^2 + 7s + 3} \tag{25}$$

For the relay system assume $H = 10$, $h = 0.005$; for the continuous system assume $k = 500$, $u_{mx} = +H = 10$, $u_{min} = -H = -10$.

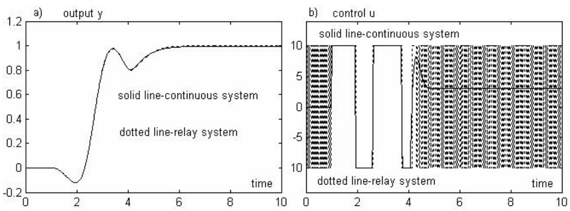

Fig. 6. Comparison of a) outputs y and b) controls u for the continuous and relay system from Example 2.

The results of simulations for reference value $r(t) = \mathbf{1}(t-1)$, for both the systems are shown in Fig. 6a and b. Solid line is used for the continuous system and dotted line – for the relay system. From Fig. 6a it is seen that in accordance with above considerations the responses of both the systems are almost the same though the controls shown in Fig. 6b are completely different. The controls for both the systems cover only in the time intervals where they take constant values 10 or -10.

8. FINAL CONCLUSIONS

In the case of minimum phase plants, for which the usual sliding mode control may be applied, it is shown how to create the equivalent continuous system, which has the same response y for the same external excitations. In the equivalent system the relay is replaced by the amplifier with high gain k connected in series with the saturation for which $u_{mx} = +H$ and $u_{min} = -H$. The equivalence takes place if the relay hysteresis is small. If not then the output with sliding mode control has oscillations and the continuous system from the point of view of the output waveform is better.

In the case of nonminimum phase, or other difficult plants, the similar equivalence has been utilized to create the system with modified sliding mode control. In this system the fast switching of the relay is obtained with using parallel compensator (Gessing, 2007) (without approximations of higher order derivatives, as in usual sliding mode control).

In the equivalent continuous systems the chattering effect appearing in sliding mode control does not exist. Then, at the first look it seems, that the sliding mode control has a weak justification to be of interest. However sometimes the simplicity of the actuator in the form of the switch implementing the characteristic of the relay, is more important than some insignificant decrease of the control caused by chattering, which may decide about application of sliding mode control.

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