USING A CHEBYSHEV APPROACH FOR THE MINIMUM-TIME OPEN-LOOP CONTROL OF CONSTRAINED MIMO SYSTEMS

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Abstract: In this paper we propose the use of a technique based on Chebyshev polynomials approximation for determining the minimum-time rest-to-rest open-loop control law for multi-input multi-output (MIMO) continuous-time systems with input and output constraints. The optimal input can be determined, without discretising the system, by suitably approximating the state variables and the input signals by means of Chebyshev series and by subsequently solving a constrained optimizsation problem. Simulation results confirm the effectiveness of the technique.

1. INTRODUCTION

It is well-known that in order to achieve a high performance in practical applications, the control system has to be designed by taking into account explicitly the (input and output) constraints of the system (Glattfelder and Schaufelberger, 2003). Actually, there are always saturation limits for the actuators and many times the outputs of the system cannot exceed given limit values. Thus, the synthesis of a minimum-time *rest-to-rest* open-loop control with input and output constraints is of great practical interest, as demonstrated also by recent research contributions. In particular, in (Consolini and Piazzi, 2006*b*) a solution for single-input single-output (SISO) continuous-time systems has been proposed. It consists in first discretising the system and then solving a sequence of linear programming problems. Thus, only an approximated solution is obtained (note also that the discrete minimum-time solution is not unique) and, most of all, the choice of the discretisation method and of the sampling time can be a critical issue (although it is demonstrated that the solution converges to the optimal continuous-time solution when the sampling time tends to zero). Remarkably, in (Consolini and Piazzi, 2006*a*) it has been demonstrated that the time-optimal control law to be determined represents a generalised bang-bang, namely, it is characterised by the fact that either the input or the output saturates during the transient.

In this paper we show that a Chebyshev approach (Vlassenbroeck, 1988; El-Gindy *et al.*, 1995; Elnagar, 1997; Elnagar and Kazemi, 1998; Jaddu and Shimemura, 1999) can be employed effectively to find the minimum-time open-loop control law for the general case of a (input and output) constrained linear continuous-time multi-input multi-output (MIMO) system. The method is based on parameterising the state variables and the control variables by Chebyshev series. In this way the system dynamics is transformed into a system of algebraic equations and therefore the minimum-time control problem is reduced into a constrained optimisation problem.

It is worth stressing that, in general, the approaches to numerical solutions of optimal control problems may be divided into two major classes: the indirect methods and the direct methods. The indirect methods are based on the Pontryagin maximum principle and often the Boundary Values Problems (BVP) which use these methods are quite difficult to solve. Conversely, direct optimisation methods convert the (infinite-dimensional) continuous problem to a finite-dimensional nonlinear programming (NLP) problem through some parameterisation of the state and control vectors. These methods can be further categorised into iterative integration methods, where only the control variables are parameterised, and collocation methods in which both the control and the state variables are parameterised (Heilig and McPhee, 1999). In this paper a direct collocation approach is therefore employed.

Note that, while the use of Chebyshev polynomials is a well-established approach for solving optimal control problems, their use has never been shown for solving the minimum-time rest-to-rest open-loop control problem for MIMO systems with constraints on both the input and output variables and for characterising the solution obtained.

Notation. The pseudoinverse of a matrix $\mathbf{M}$ is denoted as $\mathbf{M}^{\perp}$, while $[\mathbf{v}]_i$ denotes the ith component of vector $\mathbf{v}$.

2. PROBLEM FORMULATION

Although we are dealing with a system with input and output constraints and it is therefore more natural to describe it as a transfer function, it is more convenient for the subsequently applied Chebyshev optimisation to describe it in state-space form. Thus, consider the linear time-invariant continuous-time system Σ described in state-space form as

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \qquad (1)$$

where $\mathbf{A}$ is invertible, $\mathbf{x} = [x_1, \ldots, x_n]^T$, $\mathbf{u} = [u_1, \ldots, u_m]^T$ and $\mathbf{y} = [y_1, \ldots, y_p]^T$. Suppose that a transition from an initial equilibrium state $\mathbf{x^0} := \mathbf{x}(t_0)$ to a final equilibrium state $\mathbf{x^f} := \mathbf{x}(t_f)$ is required and denote the corresponding outputs as $\mathbf{y^0} := \mathbf{y}(t_0) = \mathbf{C}\mathbf{x^0}$ and $\mathbf{y^f} := \mathbf{y}(t_f) = \mathbf{C}\mathbf{x^f}$. Without loss of generality we assume hereafter null initial conditions, i.e., $\mathbf{x^0} = \mathbf{0}$ and $\mathbf{y^0} = \mathbf{0}$.

Suppose also that constraints are given for each input and each output, namely, we have

$$\begin{aligned} u_i^- \le u_i(t) \le u_i^+, \quad & i = 1, \ldots, m, \quad t \in [0, t_f] \\ y_i^- \le y_i(t) \le y_i^+, \quad & i = 1, \ldots, p, \quad t \in [0, t_f] \end{aligned} \qquad (2)$$

The time-optimal constrained open-loop control problem consists of finding the rest-to-rest transition which provides the minimum transition time t_f^* without exceeding the given constraints, namely, it consists in solving the following optimal control problem:

$$\min_{\mathbf{u}(t)} t_f \qquad (3)$$

subject to

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad 0 \le t \le t_f \qquad (4)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \quad 0 \le t \le t_f \qquad (5)$$

$$\mathbf{x^0} = \mathbf{0} \qquad (6)$$

$$\mathbf{x}(t_f) = \mathbf{x^f} \qquad (7)$$

$$u_i^- \le u_i(t) \le u_i^+, \; i = 1, \ldots, m, \; 0 \le t \le t_f \qquad (8)$$

$$y_i^- \le y_i(t) \le y_i^+, \; i = 1, \ldots, p, \; 0 \le t \le t_f \qquad (9)$$

In solving (3)-(9) we can rely on the following proposition.

Proposition 1. The optimisation problem (3)-(9) admits a solution if

$$\begin{aligned} &\{0, y_i^f\} \subset (y_i^-, y_i^+), \quad i = 1, \ldots, p \\ &\{0, [-\mathbf{B}^{\perp}\mathbf{A}\mathbf{x^f}]_i\} \subset (u_i^-, u_i^+), \quad i = 1, \ldots, m \end{aligned} \qquad (10)$$

Proof. It can be directly derived from proofs appeared in (Piazzi and Visioli, 2001) and (Consolini and Piazzi, 2006*b*).

It is worth noting that the conditions (10) simply mean that the initial and final steady-state input and output values must be included in the range allowed for the input and output signals respectively (note that $[-\mathbf{B}^{\perp}\mathbf{A}\mathbf{x^f}]_i$ is the final steady-state value of the ith component of the input vector).

The posed boundary value problem with differential-algebraic equations (BVP-DAE) can be solved with a numerical technique which consists in expanding the state and control variables in the Chebyshev series. This allows to convert the boundary conditions into algebraic equations in the unknown coefficients. In this way, the optimal control problem is replaced by a parameter optimizsation problem which consists of the minimisation of the performance index subject to algebraic constraints (Vlassenbroeck, 1988).

3. CHEBYSHEV OPTIMISATION

3.1 Chebyshev polynomials

The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_i(\tau)$. They are normalised such that $T_i(1) = 1$, $i = 0, 1, \ldots$ and in their trigonometric form they are expressed as:

$$T_i(\tau) = \cos(i \cdot \arccos(\tau)) \qquad \tau \in [-1, 1]. \qquad (11)$$

They can be also defined by the recurrence relation:

$$\begin{aligned} T_0(\tau) &= 1 \\ T_1(\tau) &= \tau \\ T_{i+1}(\tau) &= 2\tau T_i(\tau) - T_{i-1}(\tau) \quad i > 1. \end{aligned} \qquad (12)$$

In order to use the Chebyshev polynomials of the first kind for the approximation of the system dynamics, the following time transformation is therefore necessary:

$$t = \frac{t_f}{2}(1 + \tau). \qquad (13)$$

This transformation allows a change from the time domain $t \in [0, t_f]$ to the Chebyshev domain $\tau \in [-1, 1]$. The new system dynamics expressed in the Chebyshev domain is therefore (Jaddu and Shimemura, 1999):

$$\frac{d\mathbf{x}(\tau)}{d\tau} = \frac{t_f}{2}\left(\mathbf{A}\mathbf{x}(\tau) + \mathbf{B}\mathbf{u}(\tau)\right) \qquad -1 \le \tau \le 1 \qquad (14)$$

with initial and final conditions:

$$\mathbf{x}(-1) = \mathbf{x^0} = \mathbf{0}, \qquad \mathbf{x}(1) = \mathbf{x^f}. \tag{15}$$

3.2 Approximation through Chebyshev series

Once the system dynamics has been rewritten in the Chebyshev domain, the next step is the expansion of both the state vector $\mathbf{x}$ and the input vector $\mathbf{u}$ through Chebyshev series of order h:

$$\mathbf{x_h}(\tau) = \frac{1}{2}\boldsymbol{\alpha_0} T_0(\tau) + \sum_{i=1}^{h} \boldsymbol{\alpha_i} T_i(\tau) \tag{16}$$

$$\mathbf{u_h}(\tau) = \frac{1}{2}\boldsymbol{\beta_0} T_0(\tau) + \sum_{i=1}^{h} \boldsymbol{\beta_i} T_i(\tau) \tag{17}$$

where $\tau \in [-1, 1]$ and $\bar{\boldsymbol{\alpha}} := [\boldsymbol{\alpha_0}, \boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_h}]$ (with $\boldsymbol{\alpha_i} = [\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}]^T$, $i = 0, \ldots, h$) and $\bar{\boldsymbol{\beta}} := [\boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \ldots, \boldsymbol{\beta_h}]$ (with $\boldsymbol{\beta_i} = [\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}]^T$, $i = 0, \ldots, h$) are the unknown coefficients. The same order h has been assumed for both the state and the input for the sake of simplicity. The choice of h is related to the required accuracy. Actually, increasing its value yields to a better approximation (indeed, in principle, for h that tends to infinity, the approximation tends to the exact solution), but, from another point of view, increases also the complexity of the optimisation problem. The value of $h = 80$ has therefore been selected to provide a high accuracy with a reasonable computational effort.

3.3 Equality and inequality constraints

The approximation of the state and the input variables $\mathbf{x_h}$ and $\mathbf{u_h}$ is then substituted into the ordinary differential equation (14), yielding to:

$$\frac{d\mathbf{x_h}(\tau)}{d\tau} = \frac{t_f}{2}\left(\mathbf{A}\mathbf{x_h}(\tau) + \mathbf{B}\mathbf{u_h}(\tau)\right) \qquad -1 \le \tau \le 1. \tag{18}$$

as well as into the initial and final condition equations (15). The left hand side of equation (18) can be derived by considering that the derivative of the series (16) with respect to τ is given by

$$\frac{1}{2}\boldsymbol{\alpha'_0} + \sum_{i=1}^{h-1} \boldsymbol{\alpha'_i} T_i(\tau) \tag{19}$$

where the coefficients $\bar{\boldsymbol{\alpha'}} := [\boldsymbol{\alpha'_0}, \boldsymbol{\alpha'_1}, \ldots, \boldsymbol{\alpha'_{h-1}}]$ can be expressed in terms of the coefficients $\boldsymbol{\alpha}$ by means of the following formula (Fox and Parker, 1972):

$$\boldsymbol{\alpha'_{r-1}} - \boldsymbol{\alpha'_{r+1}} - 2r\boldsymbol{\alpha_r} = 0, \quad r = 1, \ldots, h-1. \tag{20}$$

Thus we obtain

$$\frac{t_f}{2}\left(\mathbf{A}\mathbf{x_h}(\tau) + \mathbf{B}\mathbf{u_h}(\tau)\right) = \frac{1}{2}\boldsymbol{\alpha'_0} + \sum_{i=1}^{h-1} \boldsymbol{\alpha'_i} T_i(\tau) \tag{21}$$

By equating the coefficients of same-order Chebyshev polynomials we obtain a system of $n \times h$ nonlinear equality constraints (note that the final time t_f is unknown).

The substitution of $\mathbf{x_h}$ into the initial and final condition expression (15) yields to $2 \times n$ additional equality constraints, namely, (Vlassenbroeck, 1988)

$$\frac{1}{2}\boldsymbol{\alpha_0} + \sum_{i=1}^{h} (-1)^i \boldsymbol{\alpha_i} - \mathbf{x}(-1) = 0 \tag{22}$$

and

$$\frac{1}{2}\boldsymbol{\alpha_0} + \sum_{i=1}^{h} \boldsymbol{\alpha_i} - \mathbf{x}(1) = 0. \tag{23}$$

The inequality constraints (8)-(9) can be handled by rewriting them as

$$\begin{array}{ll} y_i^- - [\mathbf{C}\mathbf{x_h}(\tau)]_i \le 0 & i = 1, \ldots, p \\ [\mathbf{C}\mathbf{x_h}(\tau)]_i - y_i^+ \le 0 & i = 1, \ldots, p \\ u_i^- - [\mathbf{u_h}(\tau)]_i \le 0 & i = 1, \ldots, m \\ [\mathbf{u_h}(\tau)]_i - u_i^+ \le 0 & i = 1, \ldots, m \end{array} \tag{24}$$

and by defining four vectors of slack variables $\mathbf{w_i}(\tau)$, $i = 1, \ldots, 4$. Thus, expression (24) can be rewritten as

$$\begin{array}{ll} y_i^- - [\mathbf{C}\mathbf{x_h}(\tau)]_i = -[\mathbf{w_1}(\tau)]_i^2 & i = 1, \ldots, p \\ [\mathbf{C}\mathbf{x_h}(\tau)]_i - y_i^+ = -[\mathbf{w_2}(\tau)]_i^2 & i = 1, \ldots, p \\ u_i^- - [\mathbf{u_h}(\tau)]_i = -[\mathbf{w_3}(\tau)]_i^2 & i = 1, \ldots, m \\ [\mathbf{u_h}(\tau)]_i - u_i^+ = -[\mathbf{w_4}(\tau)]_i^2 & i = 1, \ldots, m \end{array}. \tag{25}$$

At this point each slack variable can be expanded in a Chebyshev series with unknown coefficients $\bar{\boldsymbol{\gamma}}$ and by equating again the coefficients of same-order Chebyshev polynomials, a set of (nonlinear) equality constraint relations in $\bar{\boldsymbol{\alpha}}$, $\bar{\boldsymbol{\beta}}$ and $\bar{\boldsymbol{\gamma}}$ results. In this way an optimisation problem with only (nonlinear) equality constraints is obtained.

Alternatively, the Chebyshev series in (24) can be evaluated at a number of points τ_i, $-1 = \tau_0 < \tau_1 < \cdots < \tau_k = 1$, so that a set of inequality constraint relations in $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$ results. Although this approach is less rigorous, we preferred to use it because, overall, it requires a less computational effort.

3.4 Optimisation

By following the steps described before, the optimal control problem (3)-(9) is therefore transformed into a parameter optimisation problem which consists in finding t_f, $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$ in order to minimise the transition time t_f subject to the posed equality and inequality constraints. The minimum-time open-loop control law is then obtained by applying expression (17).

To solve this optimisation problem, a sequential quadratic programming (SQP) method, such as the one implemented in the function "`fmincon`" of Matlab can be used (Matlab, 2006). In this context the starting values of the parameters t_f, $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$, denoted respectively as t_f^0, $\bar{\boldsymbol{\alpha}}^0$ and $\bar{\boldsymbol{\beta}}^0$ can be selected through the Chebyshev interpolation of the state variables evolution of the system when a step signal (whose amplitude is that required at the final equilibrium point) is applied to each system input. In particular, the value of t_f^0 can be selected as the largest (2%) settling time of the outputs. Then, consider the $h+1$ Chebyshev nodes that are obtained as

$$\tau_i = \cos\left(\frac{\pi i}{h}\right) \qquad i = 0, \ldots, h. \tag{26}$$

These nodes are symmetrically distributed about $\tau = 0$. As far as i increases, they cluster towards the endpoints of the interval. If we denote by $\bar{x}_i(\tau)$ the interpolant of function $x_i(\tau)$, $i = 1, \ldots, n$ at the Chebyshev nodes, we have (Quarteroni and Valli, 1997):

$$\bar{x}_i(\tau) = \sum_{j=0}^{h} \alpha_{ji}^0 T_j(\tau) \tag{27}$$

where

$$\alpha_{ji}^0 = \frac{2}{hd_j} \sum_{k=0}^{h} \frac{1}{d_k} \cos\left(\frac{kj\pi}{h}\right) x(\tau_k) \tag{28}$$

with

$$d_i = \begin{cases} 2 & \text{for} \quad i = 0, h \\ 1 & \text{for} \quad i = 1, \ldots, h-1 \end{cases} \tag{29}$$

Having chosen step signals for the m inputs, for the Chebyshev series (17) the best initial fitting is obtained with β_{0i}^0 $(i = 1, \ldots, m)$ equal to the value of the step amplitude of the ith input multiplied by two and $\boldsymbol{\beta_1}^0 = \cdots = \boldsymbol{\beta_h}^0 = \mathbf{0}$.
Finally, it is worth stressing that the optimisation algorithm can be made faster by providing the explicit expression of the gradient of both the equality and inequality constraints with respect to t_f, $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$.

4. ILLUSTRATIVE RESULTS

As an illustrative example we consider the following system:

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{10} & 0 & 0 & 0 \\ 0 & -\frac{1}{15} & 0 & 0 \\ 0 & 0 & -\frac{1}{15} & 0 \\ 0 & 0 & 0 & -\frac{1}{10} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} \frac{3}{5} & 0 & \frac{8}{15} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{3}{5} \end{bmatrix} \tag{30}$$

Note that the system zeros are $\{5.2833, -0.0833\}$. Three different cases are considered. In the first case we selected $\mathbf{y^f} = [1\ 1]^T$, $y_1^- = y_2^- = -0.01$, $y_1^+ = y_2^+ = 1.01$, $u_1^- = u_2^- = -1$ and $u_1^+ = u_2^+ = 1$. The application of the Chebyshev methodology yields to an optimal rest-to-rest transition time $t_f^* = 54.1$ and to the system inputs and outputs shown in Figures 1 and 2 respectively. It can be seen that the constraints are not violated. Indeed, the active constraints are those on the outputs (with the exception of the first part of the transient response). It can be noted that the optimal rest-to-rest transition time is much greater than the settling times of the outputs. This is due to the fact that the state of the system takes $t_f^* = 54.1$ to attain the new equilibrium point, as can be seen in Figure 3. The values of the optimal transition time determined when the (significant) constraints are modified are shown in Figure 4. In particular, each constraint has been modified by keeping the other ones constant (for example, in the top-left figure, the value of t_f^* for different values of u_1^+ is shown). It can be seen that Pareto fronts emerge, namely, the optimal transition time decreases by relaxing one of the constraint (either on the inputs or on the outputs), as expected.

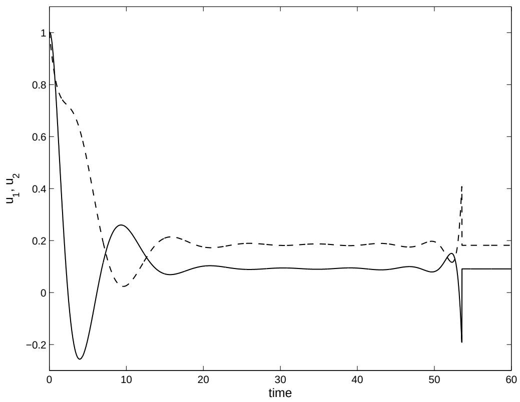

Fig. 1. System inputs for $\mathbf{y^f} = [1\ 1]^T$ (u_1: solid line; u_2: dashed line).

In the second example, with the same constraints on the inputs, we selected $\mathbf{y^f} = [1\ 0]^T$, $y_1^- = -0.01$, $y_1^+ = 1.01$, $y_2^- = -0.01$, $y_2^+ = 0.01$. Indeed, the aim of the control law is to achieve a transition for y_1 by keeping y_2 very close to the initial conditions during the transient. Optimal inputs and outputs, corresponding to $t_f^* = 56.8$ are shown in Figures 5 and 6 (the state evolution is shown in Figure 7). It can be seen that there are active constraints on both the outputs and on u_2 (in the first part of the transient). The values of t_f^* obtained by modifying one constraint at a time are shown in Figure 8.
Finally, the case where $\mathbf{y^f} = [0\ 1]^T$, $y_1^- = -0.01$, $y_1^+ = 0.01$, $y_2^- = -0.01$, $y_2^+ = 1.01$ has been considered (again, the constraints on the inputs are the same of the previous cases). In this case the Chebyshev optimisation yields to $t_f^* = 55.2$. Results are shown in Figures 9-12 where it appears that constraints on both the inputs and the outputs are activated.

5. CONCLUSIONS

In this paper we have shown that a technique based on Chebyshev polynomials can be employed effectively to find the (approximated) solution of the time-optimal constrained open-loop control problem for MIMO systems. It appears from the results presented that (at least) one of the posed

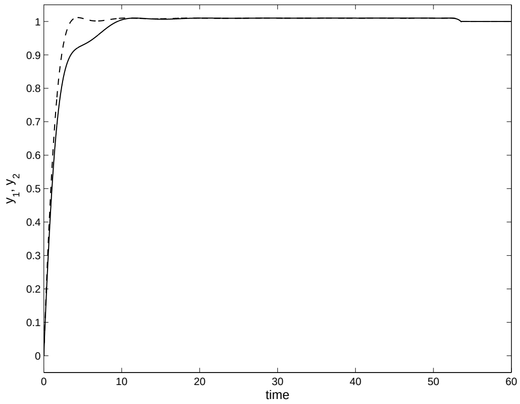

Fig. 2. System outputs for $\mathbf{y^f} = [1\ 1]^T$ (y_1: solid line; y_2: dashed line).

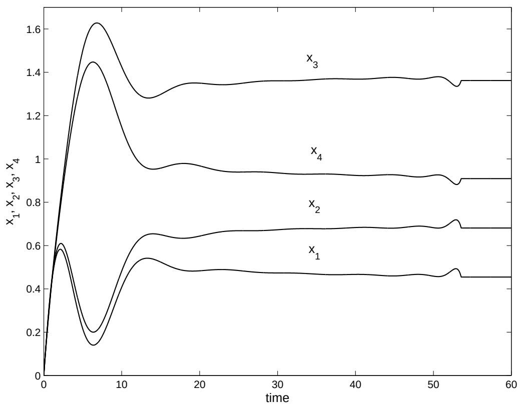

Fig. 3. System states for $\mathbf{y^f} = [1\ 1]^T$.

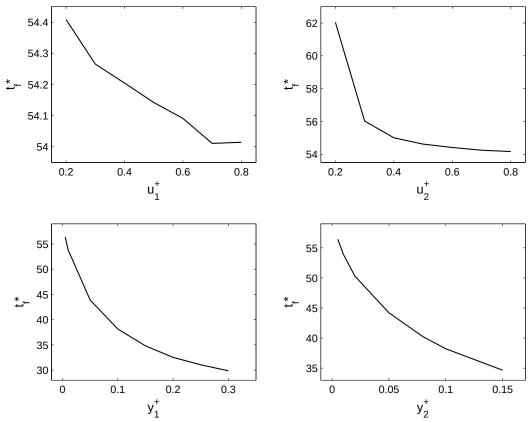

Fig. 4. Optimal transition times with modified constraints for $\mathbf{y^f} = [1\ 1]^T$.

constraints is active while the state evolves until the final equilibrium point is attained. The method can be used in practical cases to achieve a fast output transition by taking into account system constraints (note that in the proposed examples the (2%) output response settling time is much shorter than the rest-to-rest transition time).

It is worth stressing that, by selecting the values of the output constraints appropriately, the methodology can be employed effectively to achieve a practical decoupling of the system. In other words, from a practical point of view, the suitably determined feedforward control law is capable to

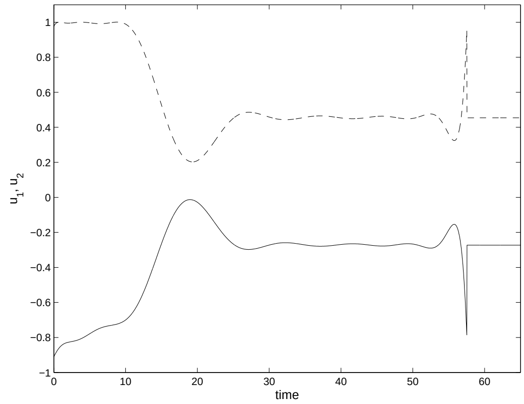

Fig. 5. System inputs for $\mathbf{y^f} = [1\ 0]^T$ (u_1: solid line; u_2: dashed line).

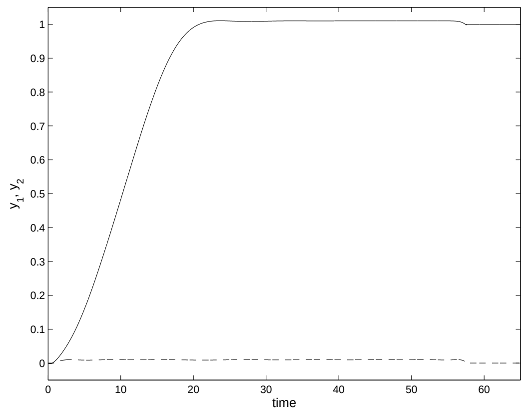

Fig. 6. System outputs for $\mathbf{y^f} = [1\ 0]^T$ (y_1: solid line; y_2: dashed line).

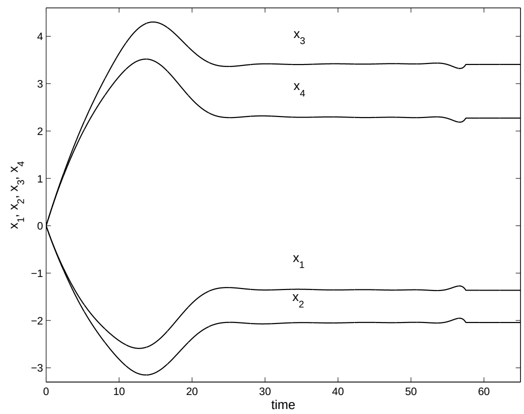

Fig. 7. System states for $\mathbf{y^f} = [1\ 0]^T$.

achieve a desired transition of one of the outputs while the others are kept at a constant value. This implies that the proposed methodology can be used in practical applications in order to substitute the presence of a decoupling controller.

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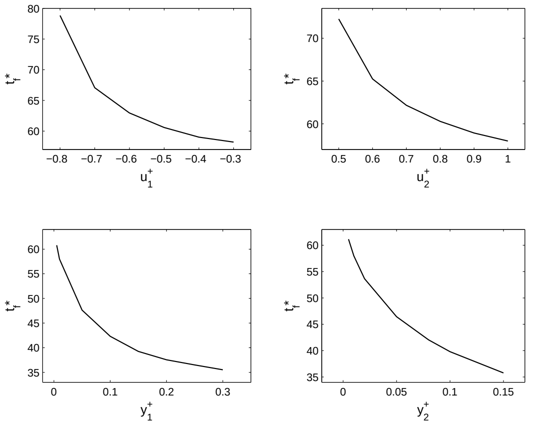

Fig. 8. Optimal transition times with modified constraints for $\mathbf{y^f} = [1\ 0]^T$.

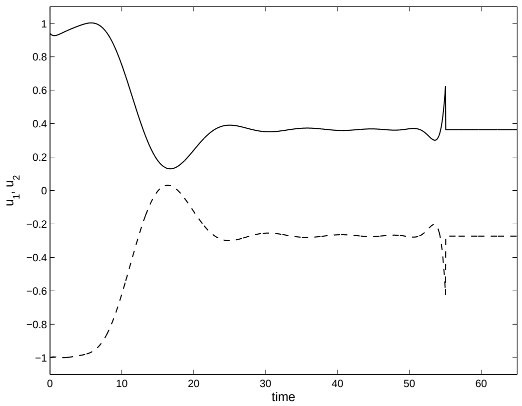

Fig. 9. System inputs for $\mathbf{y^f} = [0\ 1]^T$ (u_1: solid line; u_2: dashed line).

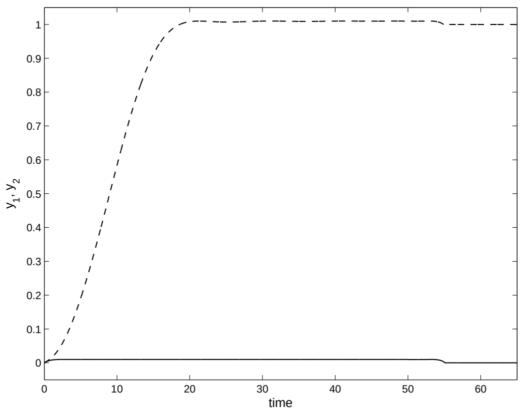

Fig. 10. System outputs for $\mathbf{y^f} = [0\ 1]^T$ (y_1: solid line; y_2: dashed line).

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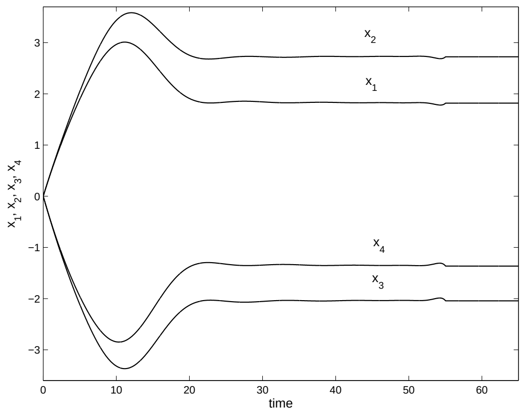

Fig. 11. System states for $\mathbf{y^f} = [0\ 1]^T$.

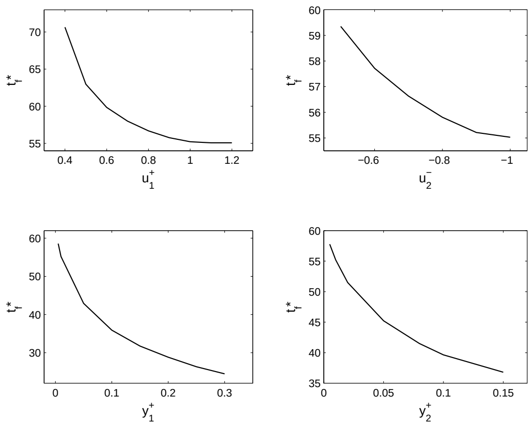

Fig. 12. Optimal transition times with modified constraints for $\mathbf{y^f} = [0\ 1]^T$.

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