STRUCTURAL ANALYSIS OF THE DEFORMATION OF THE FLEXIBLE ARM OF A ROBOT: PART 2

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Abstract: We consider linear structured systems in state space form where a linear system is structured when each entry of its matrices A, B, C and D are either a fixed zero or a free parameter. The structure of the system is determined by the location of the fixed zeros in these matrices. A structured system is representative of a class of linear systems depending on the possible values of the free parameters. The properties of structured systems are true for almost any value of these free parameters. Theses structural properties can in general be checked by means of directed graphs that can be associated to a structured system. Its vertices correspond to the input, state and output variables, and the edges between two vertices correspond to nonzero parameters relating the corresponding variables in the equation. This paper presents an illustrative application of the previous notions. It consists in analyzing and verifying some structural properties (controllability, observability and structure at infinity) of the deformation of a flexible arm of a robot under a given acceleration modelled in state space form.

Keywords: Linear systems, structured systems, structural properties, graph theory, flexible robot arm modelling.

1. INTRODUCTION

Linear systems have been studied using various approaches based on state space models, transfer matrices, matrix pencils, polynomial factorizations, and so on. The system may contain fixed parameters representing the specific role that certain variables play in the system, or fixed algebraic relations between variables and the absence of relations between variables gives rise to fixed zero entries. The system may also contain parameters that represent empirical relations between variables and that enter the system description through the physical modelling. Such parameters can be obtained by means of identification and they are subject to uncertainties and modelling errors. The usual approaches of linear systems do not allow taking into account available parametric information and they often assume full knowledge of the parameters. The notion of structured systems allows coping with some of the previous points. The study of structured systems may be considered to have been started with Lin (1974). In this paper and also in later papers (Glover & Silverman, 1976; Shields & Pearson, 1976), the controllability of structured systems is studied. The generic rank of the transfer matrix of a structured system and the generic structure at infinity has been studied during many years (Dion *et al.*, 2003).

In a previous paper (Yaici and Belmehdi, 2006), an illustrative application of the previous notions has been presented. It consists in analyzing and

verifying some structural properties (controllability, observability and structure at infinity) of the one degree flexion deformation of a flexible arm of a robot under a given acceleration modelled in state space form, neglecting dissipation and boundary conditions. This paper continues from the previous one, to introduce the linear dissipation, but the study of other possible deformations is still under investigation.
The paper starts by a recall of structural systems, graph theory and properties of linear systems. The discrete modelling of the deformation to obtain the needed state space representation under dissipation is then presented and for each case a structural analysis of the model is given. The paper finishes with a conclusion and some perspectives.

2. MATHEMATICAL BACKGROUND

In this section we recall the definition of linear structured systems; we introduce the associated graph of a structured system and give the graph characterization of the infinite structure.

2.1 Modelling of the robot arm

It is a flexible arm of a robot incrusted in a motor (figure 1). We try to model the single one

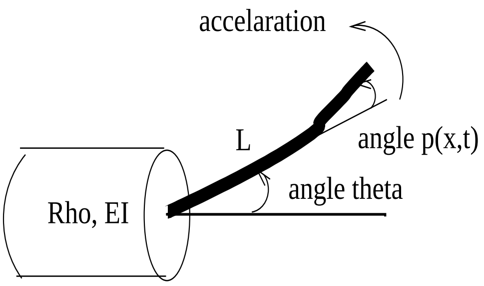

Fig. 1. The flexible arm robot

degree flexion deformation of the arm of the robot due to an acceleration, the angle θ being the angular position. To model this deformation, the arm length is sampled in the space in N points: $p(i,t)$ represents the position of the i^{th} piece of the arm in time t. The extremity of the arm, hence $p(N,t)$, must do a move from Θ_1 to Θ_2 in a delay $T(t_0 = 0, t_N = T)$. The Lagrangian of this linear displacement is as follows:

$$L = \int\int\{\frac{1}{2}\rho(\ddot{\Theta}_x + \ddot{p})^2 - \frac{1}{2}EI\frac{d^2p}{dx^2}\}dxdt \qquad (1)$$

Where $p(x,t)$ is the deformation by a one degree flexion and $\dot{p}$, $\dot{\Theta}$, $\ddot{p}$ are respectively the first time derivative of p and θ and the second space derivative of p, L the length of the flexible arm and ρ and EI are constants of the mechanics of the arm.(Rouff and Cotsaftis, 2000)

2.2 Introduction to graph theory

Graph: A graph G is a mathematical structure consisting in a set of vertices ($S = s_1, s_2, \ldots$) and a set of edges ($E = e_1, e_2, \ldots$) such that each edge has its extremities in the set S.(Reinshcke, 1988)

- We associate two applications I and $T : E \rightarrow S$ which for each edge we associate its initial extremity and its final extremity.
- Incidence: topological relations between edges and vertices and vice-versa.
- we define the mxm incident matrix MI to edges of G:

$$MI_{ij} = \begin{cases} +1 \; if \; x_i \; is \; the \; initial \\ \qquad extremity \; of \; u_j \\ -1 \; if \; x_i \; is \; the \; final \\ \qquad extremity \; of \; u_j \\ 0 \; else \end{cases}$$

- Sub-graph: If the set of vertices S of a graph G can be partitioned in two disjoint sub-sets S_1 and S_2 such that each edge of G has one extremity in S_1 and the other in S_2 so the graph G is called biparted.

Directed graph: A directed graph consists in edges with arrows. We define:

- Initial vertex: The vertex source of the edge.
- Final vertex: the vertex destination of the edge.

Path: A sequence of edges $\{e_1, , e_k\}$ such that the initial vertex of the following edge is the final vertex of the precedent edge. We define:

- Length: Number of edges in the sequence.
- Simple path: the edges which constitute the path are disjoints.

Connectivity: A graph G is said to be softly connected if for all $(x_i,\ x_j) \in S(i \neq j)$, there is a sequence joining x_i to x_j.

- The binary relation: $x_iRx_j \Longleftrightarrow \begin{cases} x_i = x_j \\ \exists \; a \; path \; joining \; x_i \; and \; x_j \end{cases}$ is an equivalence relation.
- The equivalent classes on the set of vertices of G are called the connected components of G.
- A graph may be partitioned in its connected elements.

2.3 Structured Systems

The structural analysis explores the connections in the structure of the model and the functional dependency between its elements. It is useful to understand the problem and formulate its solution even with a lack of numerical information on the model. (Dion *et al.*, 2003)

We consider a linear system Σ described by:

$$\begin{cases} \dot{x}\ (t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (2)$$

With $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ and A, B, C are real matrices with adequate dimensions. A system is structured if the matrix $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ is a structured matrix, i.e. its entries are either zeros or depend on a parameter. Let Λ be the vector of all parameters $\{i = 1,..,k\}$. A structured system with a vector of parameters is denoted: Σ_Λ.

To a structured system we can associate a directed graph $G(\Sigma_\Lambda) = (Z, W)$ where Z is the set of vertices and W is the set of directed edges: $Z = U \bigcup X \bigcup Y$ where $U = \{u_1, \ldots, u_m\}$, $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_p\}$ and $W = \{(u_i, x_j), b_{ji} \neq 0\} \bigcup \{(x_i, x_j), a_{ji} \neq 0\} \bigcup \{(x_i, y_j), c_{ji} \neq 0\}$.

The structured matrix is the matrix of incidence of its corresponding graph.

2.4 Control Problems

We consider a structured system Σ to which we associate a directed graph $G(\Sigma_\Lambda)$

2.4.1. Structural rank of a matrix (Jonhston *et al.*, 1984)

The generic rank of a structured matrix is the maximum possible rank achievable by that matrix. For non structured matrices, the row echelon form reduction is used to find its rank. For structured matrices, this method can be used to determine its generic rank. Its disadvantage is that valuable information in control system synthesis is lost.
For structural matrices represented by a digraph, a method based on the concept of dilation can be used.
Concept of dilation (Lin, 1974) Let G be a directed graph, P a set of state vertices, T a determined set of all vertices source of a directed edge going to a node in P; if the number of nodes in P is greater than the number of nodes in T, then the digraph G contains dilation. Dilation in a digraph is equivalent to rank deficiency in a structural matrix.

2.4.2. Structural controllability

For the system Σ described by equation 2, it is known that the structural pair $[A : B]$ is structurally controllable if and only if:

- Its graph is input connectable and
- The structural rank : $rank[A : B] = n$

Definition 1. A class of systems with equivalent structure is structurally controllable (s-controllable) if there exist at least one realization of this class which is completely controllable.

Definition 2. A class of systems is said to be input-connectable if in the corresponding graph G there exist for each state vertex (x) a path with at least one of the input vertices (u).

By duality the s-observability is defined and a system which is s-controllable and s-observable is called s-complete.

2.4.3. Infinite Structure (Hovelaque *et al.*, 1997)

We consider the system Σ given by equation 2, and let $T(s)$ be the rational proper pxm transfer function matrix of the system Σ defined by: $T(s) = C(sI - A)^{-1}B$. The infinite structure qualifies the way the transfer matrix $T(s)$ loses in the rank when s goes to infinity.
In the Multi variable case, the infinite structure is constituted from the list of the orders of the infinite zeros at infinity of the system. To determine the infinite structure of a system we use the Smith MacMillan factorization at infinity:

Definition 3. The proper rational transfer matrix $T(s) = C(sI-A)^{-1}B$ can be factorized as follows: $T(s) = B_1(s) \begin{bmatrix} \Delta(s) & 0 \\ 0 & 0 \end{bmatrix} B_2(s)$ where:

- $\Delta(s) = diag(s^{-n}, , s^{-n})$
- n_i: integers such that $n_1 \leq n_2 \ldots \leq n_r$
- r: rank of $T(s)$.
- $B_1(s)$ and $B_2(s)$ are bi-proper rational matrices (proper rational matrices with a proper rational inverse).

The list $(n_1, n_2, \ldots n_r)$ is uniquely defined and constitutes the infinite structure of $T(s)$. The n_i's are called the infinite zeros of the system.

Theorem 4. Let Σ_Δ be a linear structured system and $G(\Sigma_\Delta)$ its associated graph; one has the following:

- The structural rank of Σ_Δ which is the number of structural infinite zeros is equal to the maximum number of input-output vertex disjoint paths in $G(\Sigma_\Delta)$.
- The structural infinite zeros orders of Σ_Δ are characterized on $G(\Sigma_\Delta)$ as follows:

$$\begin{cases} n_1 = L_1 - 1 \\ n_k = L_k - \displaystyle\sum_{j=1}^{k-1} n_j - k \\ \quad = L_k - L_{k-1} \ for \ k = 2 \ldots r \end{cases} \quad (3)$$

Where L_k is the minimal sum of vertex disjoint input-output path lengths in $G(S_\Delta)$.

Remark 5. A directed input-output path has its first vertex in U and its last vertex in Y. A set of k input-output paths with no common vertices is called a k vertex disjoint input-output paths set.

3. STRUCTURAL MODELLING OF THE DEFORMATION

3.1 State Space Form of the Deformation

In order to use graph methods we need to model the deformation in state space equations with an eventual dissipation multiplied by a factor b and the solution of equation 1 will be written as follows:

$$\rho\ddot{p} + EIp^{(4)} + b\dot{p}^{''} = -\rho\ddot{\Theta}x \qquad (4)$$

If $p(x,t)$ is sampled on $n{+}1$ points with $\Delta pmax \leq 0.5$ then equation 4 will become :

$$\rho\ddot{p}_n + EI\sum_{l=-N}^{N} a_{n-l}p_{n-l} + \sum_{l=-N}^{N} b_{n-l}\dot{p}_{n-l} = -\rho\ddot{\Theta}x (5)$$

Using Fourier Transform to compute the "a" and "b" parameters:

$$TF\left\{p_n\right\} = \sum_{n=-\infty}^{+\infty} p_n e^{-2\pi jkn\Delta x} = P(k) \qquad (6)$$

with $k = \frac{1}{\lambda}$
$TF\{p_{n-1}\} = f(TF\{p_n\})$ implies that:

$$\begin{aligned} TF\left\{p_{n-1}\right\} &= \sum_{n=-\infty}^{+\infty} p_{n-1}e^{-2\pi jk\Delta x(n-1)} \\ &= e^{2\pi jk\Delta x}\sum_{n=-\infty}^{+\infty} p_{n-1}e^{-2\pi jkn\Delta x} \\ &= P(k)e^{-2\pi jk\Delta x} \end{aligned} \qquad (7)$$

Hence Equation 5 will be decomposed in two parts:

$$\begin{aligned} &TF\left\{\sum_{l=-N}^{N} a_{n-l}p_{n-l}\right\} \\ &= \sum_{l=-N}^{N} a_{n-l}P(k)e^{-2\pi jk\Delta x} \\ &= (2\pi jk)^4 P(k) \\ &\Longrightarrow \sum_{l=-N}^{N} a_{n-l}e^{-2\pi jk\Delta xl} = (2\pi jk)^4 \end{aligned} \qquad (8)$$

and

$$\begin{aligned} &TF\left\{\sum_{l=-N}^{N} b_{n-l}\dot{p}_{n-l}\right\} \\ &= \sum_{l=-N}^{N} b_{n-l}\dot{P}(k)e^{-2\pi jk\Delta x} \\ &= (2\pi jk)^2 \dot{P}(k) \\ &\Longrightarrow \sum_{l=-N}^{N} b_{n-l}e^{-2\pi jk\Delta xl} = (2\pi jk)^2 \end{aligned} \qquad (9)$$

Limited Development of the exponential term: The last part of equation 8 and equation 9 is developed as follows:

$$\begin{aligned} &\sum_{l=-N}^{N} a_{n-l}e^{-2\pi jk\Delta x} = a_{n-4}(1-2\pi jk\Delta x)^4 \\ &+a_{n-3}(1-2\pi jk\Delta x)^3 + a_{n-2}(1-2\pi jk\Delta x)^2 \\ &+a_{n-1}(1-2\pi jk\Delta x) + a_n(1) + a_{n+1}(1-2\pi jk\Delta x) \\ &+a_{n+2}(1-2\pi jk\Delta x)^2 + a_{n+3}(1-2\pi jk\Delta x)^3 \\ &+a_{n+4}(1-2\pi jk\Delta x)^4 = (2\pi jk)^4 \end{aligned} \qquad (10)$$

and

$$\begin{aligned} &\sum_{l=-N}^{N} b_{n-l}e^{-2\pi jk\Delta x} = b_{n-2}(1-2\pi jk\Delta x)^2 \\ &+b_{n-1}(1-2\pi jk\Delta x) + b_n(1) + b_{n+1}(1-2\pi jk\Delta x) \\ &+b_{n+2}(1-2\pi jk\Delta x)^2 = (2\pi jk)^2 \end{aligned} \qquad (11)$$

We obtain the following 'a' parameters multiplied by a factor of $(1/\Delta x^4)$:

$$\left\{\begin{array}{l} u_0 : a_{00} = -7,\ a_{01} = 12,\ a_{02} = -2, \\ \quad a_{03} = -4,\ a_{04} = 1 \\ u_1 : a_{10} = a_{12} = -2,\ a_{11} = 1,\ a_{13} = 6, \\ \quad a_{14} = -4,\ a_{15} = 1 \\ u_2 : a_{20} = a_{24} = 3,\ a_{21} = a_{23} = -2, \\ \quad a_{22} = -1,\ a_{25} = -4,\ a_{26} = 1 \\ u_3 : a_{30} = a_{36} = -2,\ a_{31} = a_{35} = 3, \\ \quad a_{32} = a_{34} = -2,\ a_{33} = 1,\ a_{37} = 1 \\ for\ i = 4\ to\ N-4\ u_i : a_{ii-4} = a_{ii+4} = \dfrac{1}{2}, \\ \quad a_{ii-3} = a_{ii+3} = -2,\ a_{ii-2} = a_{ii+2} = 3, \\ \quad a_{ii-1} = a_{ii+1} = -4,\ a_{ii} = 5 \\ u_{N-3} : a_{N-3N-7} = 1,\ a_{N-3N-6} = a_{N-3N} = -2, \\ \quad a_{N-3N-5} = a_{N-3N-1} = 3, \\ \quad a_{N-3N-4} = a_{N-3N-2} = -2,\ a_{N-3N-3} = 1 \\ u_{N-2} : a_{N-2N-6} = 1,\ a_{N-2N-5} = -4, \\ \quad a_{N-2N-4} = a_{N-2N} = 3, \\ \quad a_{N-2N-3} = a_{N-2N-1} = -2,\ a_{N-2N-2} = -1 \\ u_{N-1} : a_{N-1N-5} = 1,\ a_{N-1N-4} = -4, \\ \quad a_{N-1N-3} = 6,\ a_{N-1N-2} = a_{N-1N} = -2, \\ \quad a_{N-1N-1} = 1 \\ u_N : a_{NN-4} = 1,\ a_{NN-3} = -4,\ a_{NN-2} = -2, \\ \quad a_{NN-1} = 12,\ a_{NN} = -7 \end{array}\right. \qquad (12)$$

And the following 'b' parameters (multiplied by a factor of $1/\Delta x^2$)

$$
\begin{array}{ll}
u_0 : & b_{00} = 1, b_{01} = -2, b_{02} = 1 \\
u_1 : & b_{10} = b_{12} = -1, b_{11} = 1, b_{13} = 1 \\
u_i : & b_{ii-2} = b_{ii+2} = \dfrac{1}{2}, b_{ii-1} = b_{ii+1} = -1, \\
 & b_{ii} = 1 \; (i = 2, ...N-2) \\
u_{N-1} : & b_{N-1N-3} = 1, b_{N-1N-2} = a_{N-1N} = -1, \\
 & b_{N-1N-1} = 1 \\
u_N : & b_{NN-2} = 1, b_{NN-1} = -2, b_{NN} = 1
\end{array} \tag{13}
$$

State Equations: If we choose the state variables as follows:

$$
\begin{cases}
x_0 = u_0 \\
x_1 = \dot{u}_0 \\
\vdots \\
x_{2i} = u_i \\
x_{2i+1} = \dot{u}_i \qquad (i = 1 \dots N-1) \\
\vdots \\
x_{2N-2} = u_N \\
x_{2N-1} = \dot{u}_N
\end{cases} \tag{14}
$$

$$
\Longrightarrow
\begin{cases}
\dot{x}_0 = x_1 \\
\dot{x}_1 = -\dfrac{EI}{\rho} \displaystyle\sum_{l=0}^{4} a_{0l} u_l \\
\qquad - \dfrac{1}{\rho} \displaystyle\sum_{l=0}^{2} b_{0l} \dot{u}_l \\
\vdots \\
\dot{x}_{2i} = x_{2i+1} \\
\dot{x}_{2i+1} = -\dfrac{EI}{\rho} \displaystyle\sum_{l=i-4}^{i+4} a_{il} u_l \\
\qquad - \dfrac{1}{\rho} \displaystyle\sum_{l=i-2}^{i+2} b_{il} \dot{u}_l \ddot{\Theta} i \Delta x \\
\vdots \\
\dot{x}_{2N-2} = x_{2N-1} \\
\dot{x}_{2N-1} = -\dfrac{EI}{\rho} \displaystyle\sum_{l=N-4}^{N} a_{Nl} u_l \\
\qquad - \dfrac{1}{\rho} \displaystyle\sum_{l=N-2}^{N} b_{Nl} \dot{u}_l \ddot{\Theta} N \Delta x
\end{cases} \tag{15}
$$

We obtain the following system of state equations (shown only for 5 points):

$$
\begin{cases}
\dot{x} = \begin{bmatrix}
. & + & . & . & . & . & . & . & . & . \\
+ & * & + & * & + & * & + & . & + & . \\
. & . & . & + & . & . & . & . & . & . \\
+ & * & + & * & + & * & + & * & + & . \\
. & . & . & . & . & + & . & . & . & . \\
+ & * & + & * & + & * & + & * & + & * \\
. & . & . & . & . & . & . & + & . & . \\
+ & . & + & * & + & * & + & * & + & * \\
. & . & . & . & . & . & . & . & . & + \\
+ & . & + & . & + & * & + & * & + & *
\end{bmatrix} x \\
\quad + \begin{bmatrix} . & . & . & + & . & + & . & + & . & + \end{bmatrix}^T \ddot{\Theta} \, \Delta x \\
y = \begin{bmatrix} . & . & . & . & . & . & . & . & . & + \end{bmatrix} x
\end{cases} \tag{16}
$$

Where the '+' and the '*' represent the non nulls a_{ij} and b_{ij} elements of the matrices and which will constitute the edges of the system associated graph.

4. STRUCTURAL ANALYSIS

Associated Graph The graph G can be partitioned in two subgraphs : $G = (Z_1, W') \bigcup (Z_2, W")$where $Z_1 = \{u_1\} \bigcup \{x_1, x_3, \dots x_9\}$ and $W' = \{(u_i, x_j), b_{ji} \neq 0\} \bigcup \{(x_i, x_j), a_{ji} \neq 0\}$ with $(i, j) \in \{1, 3, \dots 9\}$, and $Z_2 = \{x_0, x_2, \dots x_8\} \bigcup \{y1\}$ and $W" = \{(x_i, x_j), a_{ji} \neq 0\} \bigcup \{(x_i, y_j), c_{ji} \neq 0\}$with $(i, j) \in \{0, 2, \dots 8\}$. The associated graph is shown on figure 2.

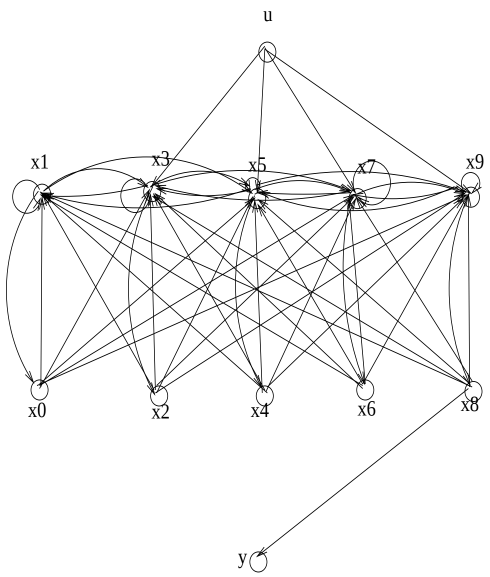

Fig. 2. Associated graph

Structural Controllability From the system associated graph we deduce that the system is structurally controllable because the graph is input connectable (figure 3) and the structural rank of [A: B] is full.

Structural Observability And the system is structurally observable because the graph is output connectable (figure 4) and the structural rank of the transpose of $[A : C]$ is full. So the properties of the s-controllability and the s-observability are verified.

infinite structure On the graph on figure 5 there exist only one disjoint input-output path, hence the number of structural zeros at infinity is one and its degree being equal to the length (which is 3) of the path minus 1 which makes 2.

5. SUMMARY

The elaborated system constitutes a primitive model of a one link robotic arm. To get a more

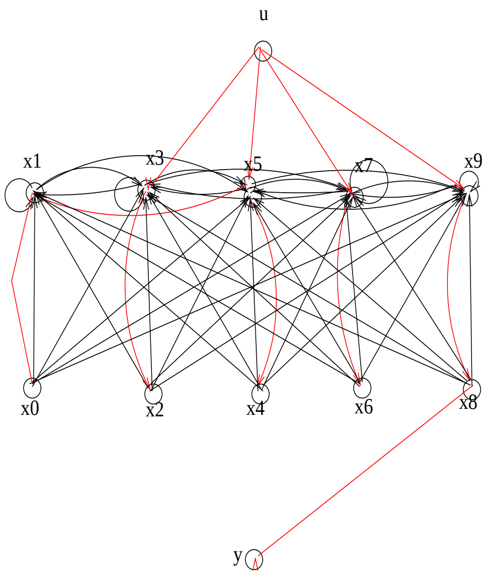

Fig. 3. input-connected graph

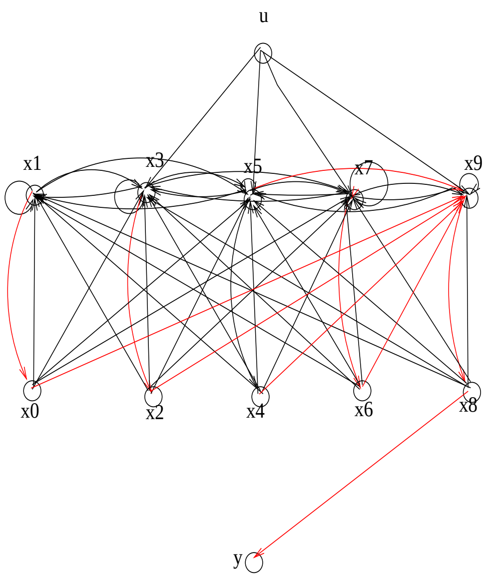

Fig. 4. output-connected graph

realistic model the sampling of the arm must respect Schannon condition ($\Delta xk \geq \frac{1}{2}$).

The paper is mainly a presentation of a method to generate automatically a linear discrete model of one link robotic arm. The model thus obtained gives normally an s-controllable and s-observable system.

The structural study of any model is advised whenever the parameters of the model are not certain or do not exist. To get the model of a deformation we can just estimate the parameters and get approximate values. So the structural study is very important here to make preliminary decisions on the model and on the system.

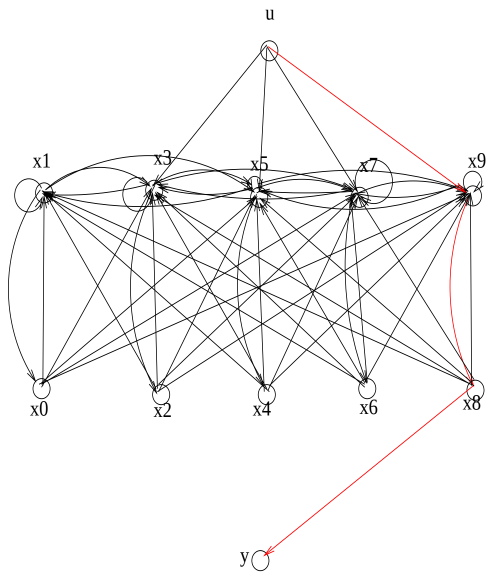

Fig. 5. input-output disjoint path graph

This paper is a first glance paper and can be taken as an example of the procedure to follow to make the structural study of any kind of complex system.

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