Properties of Output Frequencies of Volterra Systems

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Abstract: For a class of nonlinear systems, referred to as Volterra systems, some important properties for system output frequencies are studied in this paper. These properties demonstrate several novel frequency characteristics of system output spectrum and reveal clearly the nonlinear effects on system output spectrum from different kind and degree of nonlinearities. These new results have significance in the analysis and design of nonlinear systems or filters in order to achieve a specific output spectrum in a desired frequency band by taking advantage of nonlinearities, and provide an important guidance to applications of Volterra system theory in practices.

1. INTRODUCTION

Volterra systems have been extensively applied in modelling, identification, control and signal processing for many systems and engineering practices, such as circuit systems, biological systems, mechanical systems, communication systems and so on (Rugh 1981, Volterra 1959, Boyd and Chua 1985, Doyle et al 2002, Raz and Van Veen 1998, Jing et al 2008abc). The study of Volterra systems in the frequency domain was initiated by the concept of the generalized frequency response functions (GFRFs) (George 1959). Many results have been achieved for the frequency domain analysis of nonlinear systems which have a convergent Volterra series expansion (Rugh 1981, Jing et al 2006, Lang, and Billings 1996, Raz and Van Veen 1998). Nonlinear systems in the frequency domain always have very complicated output frequencies, for example, super-harmonics, sub-harmonics, inter-modulation and so on. The computation and prediction of output frequencies for Volterra systems have been studied by several authors (Raz and Van Veen 1998, Lang and Billings 1997, 2000, Bedrosian and Rice 1971, Wei et al 2007) by using the frequency domain method above. It can be seen from the past research that Volterra systems can effectively be used to account for super-harmonics and inter-modulation in the output spectrum of nonlinear systems.

In this study, some important properties for the output frequencies of Volterra systems are established. These properties reveal a straightforward and considerably complete insight into the super-harmonic and inter-modulation phenomena in the output frequencies of nonlinear systems when the systems subject to a general input function and especially the effects from different system nonlinearities are considered, and demonstrate several novel frequency characteristics of the output spectrum for nonlinear systems. These results should provide an important guidance to modelling, identification, control and signal processing by using the Volterra series theory in practices. Examples and discussions are provided to illustrate the results.

2. PROPERTIES OF OUTPUT FREQUENCIES

Volterra systems can be written as (up to order N) (Rugh 1981, Volterra 1959, Boyd and Chua 1985)

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \quad (1)$$

where $h_n(\tau_1,\cdots,\tau_n)$ is the nth-order Volterra kernel. The corresponding nth-order GFRF is defined as (George 1959)

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \quad (2)$$

The system output spectrum subject to a general input can be described as (Lang and Billings 1996)

$$Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega)$$

$$Y_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\cdot\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \quad (3)$$

$Y_n(j\omega)$ is the nth-order output spectrum. When the system input is described by

$$u(t)=\sum_{i=1}^{\bar{K}}|F_i|\cos(\omega_i t+\angle F_i) \quad (4)$$

the system output spectrum can be written as:

$$Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega)$$

$$Y_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \quad (5)$$

where $F(\omega)=\begin{cases}|F_i|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k,k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases}$.

Nonlinear systems usually have complicated output frequencies, which are quite different from linear systems. From Equations (3) and (5), the nth-order output frequencies corresponding to the nth-order output spectrum, denoted by W_n, are completely determined by $\omega = \omega_1 + \omega_2 + \cdots + \omega_n$ or $\omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}$, which produce super-harmonics and inter-modulation in system output frequencies. Therefore,

$$W_n = \left\{ \omega = \omega_1 + \omega_2 + \cdots + \omega_n \middle| \omega_i \in \overline{V}, i = 1,2,...,n \right\} \quad (6a)$$

or corresponding to the multi-tone input (4)

$$W_n = \left\{ \omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n} \middle| \omega_{k_i} \in \overline{V}, i = 1,2,...,n \right\} \quad (6b)$$

and the whole system output frequencies, denoted by W, can be written as

$$W = W_1 \cup W_2 \cup \cdots \cup W_N \quad (6c)$$

where $\overline{V}$ =-$V \cup V$ representing the input frequency range for the nth-order output spectrum, and V is the original input frequency range corresponding to the input signal, which is any continuous or discontinuous closed set in real. For example, when the system subjects to a general input, V may be a real set [a,b] $\cup$ [c,d], thus $\overline{V}$ =[-d,-c] $\cup$ [-b,-a] $\cup$ [a,b] $\cup$ [c,d], where $d \ge c \ge b \ge a > 0$. For the multi-tone input, it is a special case of the continuous one. Note that the case for V=[a,b] has been studied in (Raz and Van Veen 1998, Lang and Billings 1997), and the multi-tone case was also studied in (Lang and Billings 2000, Wei et al 2007). The study in this section is to develop some general properties for the output frequencies of system (1) in the case that V is any closed set in real. These properties can effectively demonstrate nonlinear output frequency characteristics and reveal a novel insight into the output frequencies of Volterra systems.

Property 1. The (n-2)th order output frequencies W_{n-2} are completely included in the nth order output frequencies W_n, *i.e.*, $W_{n-2} \subseteq W_n$.

Proof. Consider Equation (6a), if let $\omega_{n-1} \equiv -\omega_n$, then

$$\begin{aligned} W_n &= \left\{ \omega = \omega_1 + \omega_2 + \cdots + \omega_n \middle| \begin{array}{l} \omega_i \in \overline{V}, \omega_{n-1} = -\omega_n, \\ i = 1,2,...,n \end{array} \right\} \\ &= W_{n-2} \\ &= \left\{ \omega = \omega_1 + \omega_2 + \cdots + \omega_{n-2} \middle| \omega_i \in \overline{V}, i = 1,2,...,n-2 \right\} \end{aligned}$$

Therefore, $W_{n-2} \subseteq W_n$. The same conclusion also holds for Equation (6b). □

Property 1 can greatly facilitate the computation of output frequencies for nonlinear systems. That is, only the highest order in odd number and the highest order in even number, to which the corresponding GFRFs are not zero, are needed to be considered in Equation (6c).

Property 2. The frequencies in W_n are symmetric with respect to 0. That is, $\forall \Omega \in W_n$, then $-\Omega \in W_n$.

Proof. If $\Omega \in W_n$, then $\Omega = \omega_1 + \omega_2 + \cdots + \omega_n$. Note that $\overline{V}$ is symmetric with respect to 0, thus it must hold that $-\omega_1, -\omega_2, \cdots - \omega_n \in \overline{V}$. Therefore,

$$-\Omega = -\omega_1 - \omega_2 - \cdots - \omega_n \in W_n \quad \square$$

Property 2 shows that the negative frequencies in W are completely the same in absolute values as the positive frequencies in W. Let max(.) denote the maximum value of the elements in (.), and min(.) the minimum value.

Property 3. The maximum value of the frequencies in W_n is $n \cdot \max(V)$, *i.e.*, $\max(W_n) = n \cdot \max(V)$; and the minimum value in W_n is $-n \cdot \max(V)$, *i.e.*, $\min(W_n) = -n \cdot \max(V)$.

Proof. This is obvious from $\omega = \omega_1 + \omega_2 + \cdots + \omega_n$ and $\omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}$. □

Properties 1-3 provide some general characteristics for the output frequencies of system (1) subject to any input frequencies (i.e., V can be any closed set in real). These results can be verified by the results in (Lang and Billings 1997, 2000) where the input frequencies is [a,b] or multi-tone case. The following proposition demonstrates a novel property for the output frequencies of Volterra systems, and provides a new insight into the system output frequency characteristics.

Proposition 1 (Periodicity). The frequencies in W_n can be generated periodically as follows

$$W_n = \bigcup_{i=1}^{\Gamma_n + 1} \Pi_i(n) \quad (7a)$$

$$\Pi_i(n) = \left\{ \omega = \omega_1 + \omega_2 + \cdots + \omega_n \middle| \begin{array}{l} \omega_j \in \overline{V}, \omega_j < 0 \\ \text{for } 1 \le j \le i-1, \\ \omega_j > 0 \text{ for } j \ge i \end{array} \right\} \quad (7b)$$

or

$$\Pi_i(n) = \left\{ \omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n} \middle| \begin{array}{l} \omega_{k_j} \in \overline{V}, \omega_{k_j} < 0 \\ \text{for } 1 \le j \le i-1, \\ \omega_{k_j} > 0 \text{ for } j \ge i \end{array} \right\} \quad (7c)$$

$$\Gamma_n = n \quad (7d)$$

The above process has the following properties

$$\max(\Pi_i(n)) = -(i-1)\min(V) + (n-i+1)\max(V) \quad (8a)$$

$$\min(\Pi_i(n)) = -(i-1)\max(V) + (n-i+1)\min(V) \quad (8b)$$

$$\begin{aligned} &\max(\Pi_{i-1}(n)) - \max(\Pi_i(n)) \\ &= \min(\Pi_{i-1}(n)) - \min(\Pi_i(n)) = \min(V) + \max(V) \end{aligned} \quad (8c)$$

$$\begin{aligned} &\Delta(n) = \max(\Pi_i(n)) - \min(\Pi_i(n)) \\ &= n \cdot (\max(V) - \min(V)) \end{aligned} \quad (8d)$$

Especially, when the system subjects to a general input $U(j\omega)$ defined in [a,b] or the multi-tone input (4) with $\omega_{i+1} - \omega_i = const > 0$ for i=1,…, $\overline{K}$ -1,

$$\Pi_i(n) = \Pi_{i-1}(n) - T \quad \text{for i=2,…, n+1} \quad (8e)$$

where $\Pi_i(n) - T$ is a set whose elements are the elements in $\Pi_i(n)$ minus T, $T = \min(V) + \max(V)$ is referred to as the frequency generation period, and $\Delta(n)$ is referred to as the frequency span in each period. □

The proof is omitted. From Proposition 1 the following properties can be summarized.

Property 4. Consider the case that the system input is the multi-tone function (4). For any two frequencies Ω and Ω' in $\Pi_i(n)$ and any two frequencies ω and ω' in V, $\min(\Omega - \Omega') = \min(\omega - \omega')$. □

Property 5. When $\Delta(n)$ >T, there is overlap between the successive periods of frequencies in W_n, *i.e.*, $\max(\Pi(n)_{i+1}) > \min(\Pi(n)_i)$ for i=1,…,Γ_n.

Proof. Note that $\max(\Pi(n)_{i+1}) = \max(\Pi(n)_i) - T$, thus it can be derived,

$$\max(\Pi(n)_{i+1}) - \min(\Pi(n)_i) = \max(\Pi(n)_i) - \min(\Pi(n)_i) - T = \Delta(n) - T > 0$$ □

Proposition 1 and Properties 4-5 demonstrate an interesting and useful nature of the output frequencies. These properties can not only greatly simplify the computation of the output frequencies for some special cases as stated in Proposition 1 (where only one period of output frequencies need to be computed) but also make light for the computation of the output frequencies in general case. Especially, one of the significance of the results in this section may also be that the properties above reveal a new insight into the output frequency characteristics of nonlinear systems, *i.e.*, the periodicity. This provides a novel approach to the understanding of the nonlinear output frequency response behaviour for Volterra systems. From Proposition 1, the following corollary is straightforward.

Corollary 1. All the conclusions in Proposition 1 and Properties 1-5 hold for the case that the system subjects to a general input $U(j\omega)$ defined in $\bigcup_{i=1}^{Z}[a + (i-1)\varepsilon, b + (i-1)\varepsilon]$ where $b \ge a, \varepsilon \ge (b-a)$ and Z is a positive integer.

Note that when V does not satisfy the condition in Corollary 1, the property in Equation (8e) can not hold. In order to illustrate the results above, Example 1 is given as follows.

Example 1.Consider a simple nonlinear system as follows

$$y = -0.01\dot{y} + au^2 + bu^3$$

The input is a multi-tone function u(t)=sin(6t)+sin(7t)+ sin(8t). The output spectra are given in Figures 1-2 for different cases. Note that there are only input nonlinearities with order 2 and 3 in the system, thus only the 1st, 2nd and 3rd order GFRFs are not zero and all the other orders of the GFRFs are zero (Jing et al 2006). Hence, the nonlinear output frequencies of the system are the same as the 2nd and 3rd order output frequencies. That is, when *a*=1 and *b*=0, then $W=W_2$; when *a*=0 and *b*=1, then $W=W_3$; and when *a*=1 and *b*=1, then $W = W_2 \cup W_3$. Figures 1-2 demonstrate clearly the results in Properties 3-4 and Proposition 1, and also show that the system output frequencies are simply the accumulation of all the output frequencies corresponding to each order output spectrum when the involved nonlinearities have no crossing effect and no overlap as stated in Property 5. When and how there are crossing effects between different nonlinearities will be discussed in the next section.

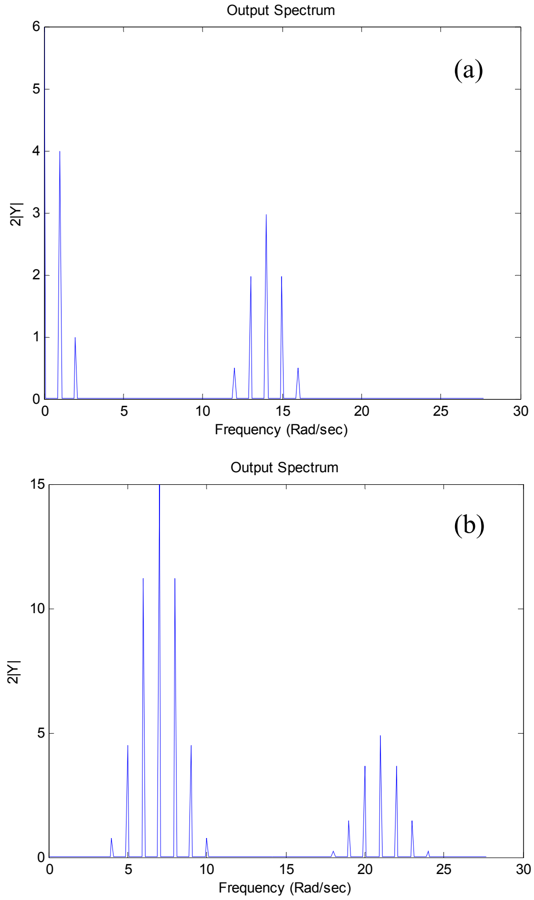

Figure 1. Output frequencies when a=1 and b=0 (a) and when a=0 and b=1 (b)

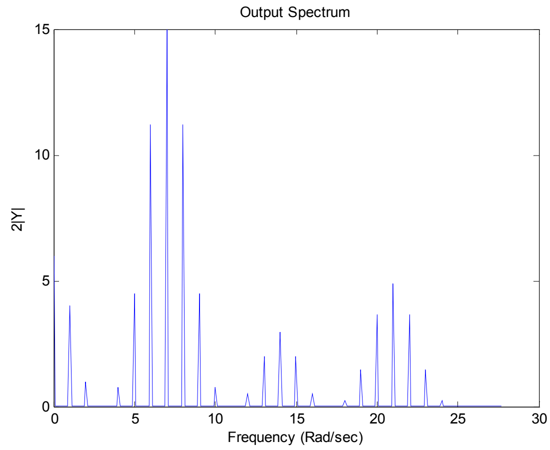

Figure 2. Output frequencies when a=1 and b=1

3. PROPERTIES OF THE OUTPUT FREQUENCIES BASED ON PARAMETRIC CHARACTERISTIC ANALYSIS

There are different kinds of nonlinearities. How a nonlinear term affects system output frequencies and what the effect is for Volterra systems are a very interesting and important topic. However, few results have been reported for this. This section provides some useful results for this topic based on the properties developed above. Consider Volterra systems described by a nonlinear differential equation (NDE)

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{l_1,l_{p+q}=0}^{K} c_{p,q}(l_1,\cdots,l_{p+q})\prod_{i=1}^{p}\frac{d^{l_i}y(t)}{dt^{l_i}}\prod_{i=p+1}^{p+q}\frac{d^{l_i}u(t)}{dt^{l_i}}=0 \quad (9)$$

where $\left.\frac{d^l x(t)}{dt^l}\right|_{l=0}=x(t)$, $p+q=m$, $\sum_{l_1,l_{p+q}=0}^{K}(\cdot)=\sum_{l_1=0}^{K}\cdots\sum_{l_{p+q}=0}^{K}(\cdot)$, M is the maximum degree of nonlinearity in terms of y(t) and u(t), and K is the maximum order of the derivative. In this model, the parameters such as $c_{0,1}$(.) and $c_{1,0}$(.) are linear parameters, and $c_{p,q}(\cdot)$ for p+q>1 are nonlinear parameters. p+q is called the nonlinear degree of the nonlinear parameter $c_{p,q}(\cdot)$. In this section, how the model parameters affect the output frequencies of Volterra systems described by a NDE model is studied.

From Equations (3, 5), the nth-order output frequencies are also determined by the nth order GFRF. If the nth order GFRF is zero, then W_n=[]. It is known from (Jing et al 2006) that the nth order GFRF is dependent on its parametric characteristics. Thus Equations (6a-b) can be written as

$$W_n=\left\{\begin{matrix}\omega=(\omega_1+\omega_2+\cdots+\omega_n)\\ \cdot(1-\delta(CE(H_n(\omega_1,\cdots,\omega_n))))\end{matrix}\middle|\,\omega_i\in\overline{V}, i=1,2,...,n\right\} \quad (11a)$$

and

$$W_n=\left\{\begin{matrix}\omega=(\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_n})\\ \cdot(1-\delta(CE(H_n(\omega_{k_1},\cdots,\omega_{k_n}))))\end{matrix}\middle|\,\omega_{k_i}\in\overline{V}, i=1,2,...,n\right\} \quad (11b)$$

where $\delta(x)=\begin{cases}1 & x=0 \text{ or } 1\\ 0 & \text{else}\end{cases}$, CE(.) is a coefficient extraction operator defined in [15], and $CE(H_n(.))$ can be recursively determined by

$$\begin{aligned}&CE(H_n(j\omega_1,\cdots,j\omega_n))\\ &=C_{0,n}\oplus\left(\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}C_{p,q}\otimes CE(H_{n-q-p+1}(\cdot))\right)\\ &\quad\oplus\left(\mathop{\oplus}_{p=2}^{n}C_{p,0}\otimes CE(H_{n-p+1}(\cdot))\right)\end{aligned} \quad (12)$$

with the terminating condition $CE(H_1(.))$=1, where "$\oplus$" and "$\otimes$" are two basic operations for the operator CE (Jing et al 2006). In Equations (11ab), suppose W_n is empty when $\delta(CE(H_n(.)))=1$.

There are three kinds of nonlinearities in model (9) and model (10): input nonlinearity with coefficient $c_{0,q}$(.) (q>1), output nonlinearity with coefficient $c_{p,0}$(.) (p>1), and input output cross nonlinearity with coefficient $c_{p,q}$(.) (p+q>1 and p>0) (where p and q are integers). Different kind and degree of nonlinearity in a system can bring different output frequencies to the system. Equations (11a-b) demonstrate the parametric characteristics of the output frequencies for Volterra systems described by (9) and (10), by which the effect on the system output frequencies from different nonlinearities can be studied. Since negative output frequencies are symmetrical with positive output frequencies with respect to zero (Property 2), thus for convenience only non-negative output frequencies are considered in what follows.

Property 6. Regarding nonlinearities of odd and even degrees,

(a) when there are no nonlinearities of even degrees, the output frequencies brought by the nonlinearities with odd degrees happen at central frequencies $(2l+1)T/2$ for l=0,1,2,… with certain frequency span;

(b) when there are only input nonlinearities of even degrees, the output frequencies happen at central frequencies $l\cdot T$ for l=0,1,2,… with certain frequency span;

(c) in other cases, the output frequencies happen at central frequencies $l\cdot T/2$ for l=0,1,2,… with certain frequency span.

The frequency span is $\Delta(n)$ corresponding to the nth order output frequencies if applicable. □

The proof is omitted. Property 6 shows that odd degrees of nonlinearities bring quite different output frequencies to the system from those brought by even degrees of nonlinearities.

Property 7. Regarding different kinds of nonlinearities,

(a) when there are only input nonlinearities of the largest nonlinear degree n, the non-negative output frequencies are in the closed set [0, $n\cdot\max(V)$];

(b) in other cases, the output frequencies span to infinity.

Proof. (a) From Equation (12) or Proposition 2 in (Jing et al 2006), only the GFRFs of orders equal to the nonlinear degrees of the non-zero input nonlinearities are not zero since there are no other kind of nonlinearities in the system. Thus the largest order of non-zero GFRFs is n. The conclusion is therefore straightforward from Property 3. (b) If there are other kinds of nonlinearities, the largest order of nonzero GFRFs will be infinite, because for any parameter $c_{p,q}$(.) with p>0 and p+q>1, it can form a monomial with any high nonlinear degree ($c_{p,q}(.)^n$) and thus contribute to any high order GFRF from Proposition 2 in (Jing et al 2006). Thus the output frequencies can span to infinity. This completes the proof. □

Input nonlinearities can independently bring output frequencies to the system in a finite band width.

Property 8. Regarding different kinds and degrees of nonlinearities,

(a) when there are only input nonlinearities, a nonlinear term of degree n can only bring output frequencies W_n, and there are no crossing effect on output frequencies between different degrees of input nonlinearities;

(b) in other cases, a nonlinear term of degree n contributes to not only output frequencies W_n but also some high order output frequencies W_m for $m > n$ due to crossing effect with other nonlinearities.

Proof. (a) Considering a nonlinear term $c_{0,n}(.)$, it can be obtained from Equation (12) that only $CE(H_n(.))$ is not zero if the other degree and kind of nonlinear parameters are zero. That is, $c_{0,n}(.)$ only contributes to $H_n(.)$ in this case. If there are other input nonlinearities, it can be known from Proposition 2 in (Jing et al 2006) that only nonlinear parameters from input nonlinearities can not form an effective monomial which is an element of any order GFRF. That is there are no crossing effects between different degrees of input nonlinearities. (b) When there are output or input-output cross nonlinearities, it can be seen from Proposition 2 in (Jing et al 2006) that there are crossing effects between different nonlinearities, and the nonlinear degree of any effective monomial (e.g. $c_{1,q}(.)c_{0,q}(.)^n$ (q>1)) formed by the coefficients from the crossing nonlinearities can be infinity. Thus a nonlinear parameter of degree n, for example $c_{0,n}(.)$, has contribution not only to $H_n(.)$, but also to some higher order GFRFs, for example $c_{1,n}(.)c_{0,n}(.)^z$ is an element of $CE(H_m(.))$ where m=zn+n+1-z. This completes the proof. □

From Property 8, the crossing effect happens easily between the nonlinearities from output nonlinearities and input-output cross nonlinearities.

Properties 6-8 provide some novel and interesting results about the output frequencies for nonlinear systems when the effects from different nonlinearities are considered, based on the results from parametric characteristic analysis in (Jing et al 2006). Property 6 shows that odd degrees of nonlinearities have quite different effect on system output frequencies from even degrees of nonlinearities. Especially, it is shown from the properties above that input nonlinearities have special effect on system output frequencies compared with the other kinds of nonlinearities. That is, input nonlinearities can move the input frequencies to higher frequency bands without interference between different frequency generation periods. These properties may have significance in design of nonlinear systems for some special purposes in practices. For example, some proper input nonlinearities can be used to design a nonlinear filter such that input frequencies are moved to a place of higher frequency or lower frequency as discussed in (Billings and Lang 2002). The results in this section have also significance in modelling and identification of nonlinear systems. For example, if a system has only odd output frequencies when subject to a sinusoidal signal, the system may have only nonlinearities of odd degree according to Property 6. Obviously, the results in this section provide a useful guidance to the structure determination and parameter selection for the design of novel nonlinear filters and also for system modelling or identification.

Example 2. Consider a simple nonlinear system as follows

$$y = -0.01\dot{y} + au^5 - by^3 - cy^2 + u$$

The input is a multi-tone function u(t)=sin(6t)+sin(7t)+sin(8t). The output spectra under different parameter vales are given in Figures 3-5, which demonstrate the results in Properties 6-8. For the input nonlinearity, the readers can also refer to Figures 1-2.

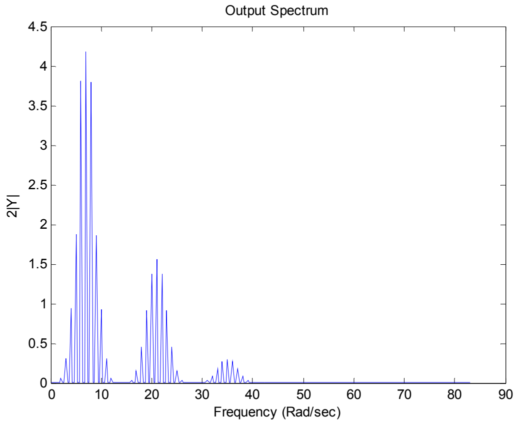

Figure 3. Output frequencies when a=0.1, b=0, c=0

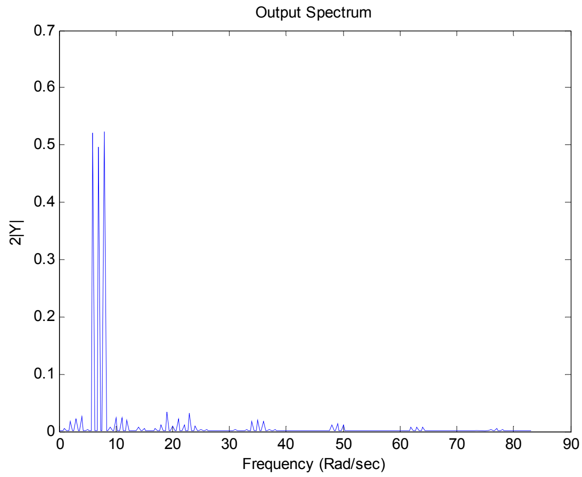

Figure 4. Output frequencies when a=0.1, b=5, c=0

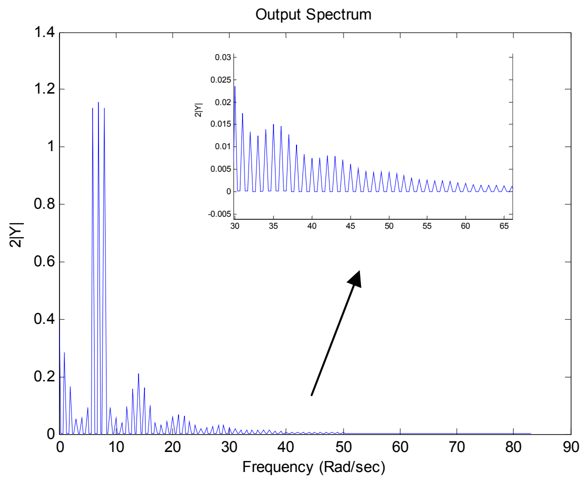

Figure 5. Output frequencies when a=0, b=0, c=0.09

When there are only odd nonlinearities, the output frequencies happen at around central frequencies 7*(2k+1). When there are even nonlinearities, the output frequencies appear at around central frequencies 7*k. The input nonlinearities only bring independently a finite band width of output frequencies to the system. The periodicity of the output frequencies can also be seen clearly from these figures.

Especially, it is worthy pointing out from Figures 1-3 that there can be no crossing effects between proper chosen input nonlinearities as mentioned before, which can not be realized by the other kinds of nonlinearities. Thus the input frequencies can be moved to higher frequency periodically without interference between different periods and then decoded by using some methods. This property may have significance when a system is designed to achieve a special output spectrum at a desired frequency band in practices by using nonlinearities.

4. CONCLUSIONS

Some interesting properties for the output spectrum of Volterra systems have been established in this paper. These properties demonstrate a novel insight into the nonlinear behaviour in output spectrum of Volterra systems, and reveal clearly the nonlinear effects on system output spectrum from different kind and degree of nonlinearities. These novel results can be used for the design of a system or filter to achieve a special output spectrum in a desired frequency band by taking advantage of nonlinearities, and provide an important and significant guidance to the analysis and design of nonlinear systems in the frequency domain by using the existing theory for Volterra systems. Further study will focus on the application issues.

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