A Design-Orientated Approach to the Geometry of Fundamental Design Limitations

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Abstract: There exist fundamental design limitations which explain why some designs are unattainable for the case that the performance variable is measured for feedback. For the case that performance variable is not measured it is found that new tradeoff can arise and this tradeoff is shown in this paper to have a nice geometry. The geometry, in addition to demonstrating the tradeoff, also renders itself as a design methodology. This ultimately results in a new perspective towards control design. This is remarkable since the conventional theory of fundamental design limitations explains the failure of some control designs but does not provide a specific design methodology. The design procedures are summarized for harmonic control as well as broad band control.

1. INTRODUCTION

It is well know that there exist fundamental performance limitations in control system designs simply because of the choice of feedback control structure. Most of these design limitations can be explained through Bode's sensitivity integral relations that have been extended to various cases, e.g. for the general case of open-loop stable/unstable continuous-time systems (Freudenberg and Looze 1985, 1987, 1988), for the case of discrete-time systems (Mohtadi 1990, Sung and Hara 1988, Maciejowski 1990) and for the case of multivariable systems (Chen 1995, 1998). Because of the fundamental significance and implications in feedback control, there has been an expansion of interest in new perspectives and aspects; the reader is referred to the IEEE Special Section on New Developments and Applications in Performance Limitations of Feedback Control (2003). A new direction arises from the fact that, as pointed out in (Freudenberg et al. 2003), most work on fundamental design limitations assumes implicitly that the system output measured for feedback is also the performance variable. In many engineering applications, especially in active noise and vibration control area, this assumption is often not feasible (Marcopoli et al. 2002, Toochinda et al. 2001, Daley and Wang 2007) since, referring to the general control configuration (Skogestad and Postlethwaite 1996) illustrated in Fig. 1, not only the performance response of exogenous input w to measurement y but also the performance response of exogenous w to performance z should be minimized. While solutions to the latter problem are available and can be found in (Zames 1981, Zames and Bensoussan 1983, Zames and Francis 1983) on sensitivity minimization, it is not widely known (Freudenberg et al. 2003) that tradeoffs exist not only between disturbance attenuation and stability robustness, but also between performance variables y and z. This is indeed the case as having been shown in (Freudenberg et al. 2003, Hong and Bernstein 1998, Bernstein 2002). However those results are again articulated mainly as Bode and Poisson sensitivity integrals. These integrals although demonstrate the tradeoffs and hence force explain the fundamental limitations in performance they do not provide us with a specific design methodology.

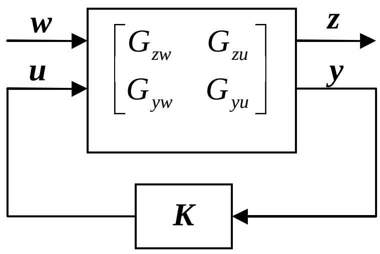

Fig. 1: The general control configuration involves four different kinds of signals: exogenous input *w*, controlled input *u*, performance *z* and measurement *y*. *K* is a stabilizing controller.

Surprisingly, as is shown in the following, there actually exists an underlying geometry which can nicely demonstrate the tradeoffs embodied ultimately in Bode's integrals but now illustrated in terms of geometric expressions. Some of the fundamental limitations are evident even at first sight. Even more remarkable is that this geometry does render itself as a design methodology and a design freedom available to the designer can be defined which possesses important properties. When this geometry is applied to broad band control, it eventually results in a new perspective towards control design—a design freedom is not necessarily the performance variable to be directly designed (compensator, for example) or minimized (sensitivity, for example) but a variable which can facilitate the overall design. The disadvantage of this geometric approach is that it currently can only deal with scalar case or the signals in Fig. 1 are scalars, as also assumed in (Freudenberg et al. 2003) and (Hong and Bernstein 1998). We will retain this assumption through this paper.

Referring to Fig. 1, the performance response of y to w is given by the transfer function:

$$T_{yw} = (1 - G_{yu}K)^{-1}G_{yw}. \quad (1)$$

The performance response of z to w is:

$$T_{zw} = G_{zw} + G_{zu}K(1 - G_{yu}K)^{-1}G_{yw}. \quad (2)$$

If we define the feedback sensitivity by

$$S = (1 - G_{yu}K)^{-1}, \quad (3)$$

and disturbance response ratio by

$$R = 1 + \frac{G_{zu}G_{yw}}{G_{zw}G_{yu}}(S - 1), \quad (4)$$

assuming for the moment that $\frac{G_{zu}G_{yw}}{G_{zw}G_{yu}}$ exists. Then (1) and (2) can be expressed as:

$$T_{yw} = SG_{yw}, \quad (5)$$

$$T_{zw} = RG_{zw}. \quad (6)$$

Hence S and R are the ratio of the closed-loop to the open-loop response with respect to the exogenous input w. We shall refer T_{yw} and T_{zw} as the closed loop disturbance response following (Freudenberg et al. 2003) for ease of reference. If we now also define the following ideal disturbance attenuation problem:

Definition: The Ideal Disturbance Attenuation Problem is solvable if for $\varepsilon, \sigma > 0$, there exists a proper and stabilizing controller such that

$$|R(j\omega)| < \varepsilon \;\&\; |S(j\omega)| < \sigma \quad \forall \omega, \quad (7)$$

then the problem of fundamental design limitations under consideration is how the control designs are inherently constrained such that the Ideal Disturbance Attenuation Problem is unsolvable. Towards this purpose, we shall investigate first how R and S, depending on the form and structure of the plants G_{zw}, G_{zu}, G_{yw}, and G_{yu}, can (possibly) compromise. This is the topic of Section 2. Such a tradeoff no matter exists or not must be subject to the controller to be proper and stabilizing, or even strongly stabilizing (since a stabilizing controller may itself not be stable (Liu and Daley 1998)). It is thus conceived that a control design in this framework is the design of a parameter which encodes the (possible) tradeoff between feedback sensitivity S and disturbance response ratio R and which also makes the control system internally stable. This point of view towards control design is elaborated in Section 3. In Section 4 design procedures are briefly summarized with this newly obtained perspective. Section 5 concludes this paper with a brief comment on previous work.

2. FUNDAMENTAL TRADEOFFS BETWEEN S AND R POSED BY SYSTEM PLANT

2.1 Geometry of S and R

From (5) and (6) it is clear that the condition for the closed loop disturbance responses T_{yw} and T_{zw} to be less than the open loop disturbance responses G_{yw} and G_{zw} for a frequency ω is

$$|S(j\omega)| < 1, \quad (8)$$

and

$$|R(j\omega)| < 1. \quad (9)$$

respectively.

The crucial point towards the geometry of S and R arises from the observation that (4) defines a Möbius transformation if S and R as defined in (8) and (9) are viewed as two circles on the complex *S-Plane* and *R-Plane* respectively. For ease of exposition, call them *unit S-circle* and *unit R-circle* on the corresponding complex planes. Under this Möbius transformation, the *unit R-circle* is mapped onto the complex *S-Plane* as another circle—call it *R-circle* with centre $(1 - real(G), -imag(G))$ and radius $|G|$, where G is defined as

$$G = \frac{G_{zw}G_{yu}}{G_{zu}G_{yw}}. \quad (10)$$

This geometry is illustrated in Fig. 2. It is obvious that *R-circle* always intersects *unit S-circle* at *point (1, 0)*, and this implies that at any arbitrary frequency this point provides non-enhancement of .the closed loop disturbance response. We shall elaborate this intersection property further in the following discussion.

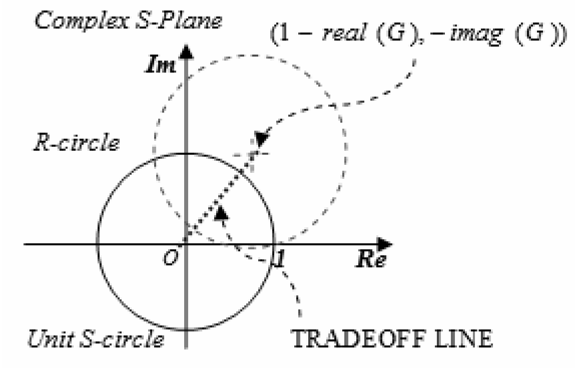

Fig. 2: Geometry of S and R

2.2 Algebraic[1] Tradeoffs between S and R

From the above construction and the properties of the Möbius mapping it can be seen that the circles $|S(j\omega)| < \sigma$ and $|R(j\omega)| < \varepsilon$ on the complex *S-Plane* are exactly the scaling with respect to the *unit S-circle* and *R-circle* respectively. This is extremely important since, on one hand the line connecting the origin and the center of the *R-circle* can be seen to define exactly the algebraic tradeoffs between S and R at a frequency ω since, referring back to Fig. 2, with the points chosen from the origin to the centre of the R-circle along the Tradeoff Line, the attenuation of R always accompanies simultaneously the enhancement of S.

On the other hand this also implies that if the two circles $|S(j\omega)| < \sigma$ and $|R(j\omega)| < \varepsilon$ intersect on the complex *S-Plane* then the points in the intersection satisfy $|S(j\omega)| < \sigma$ and $|R(j\omega)| < \varepsilon$ simultaneously. This implies that the Ideal Disturbance Attenuation Problem is *necessarily* solvable. We thus have the following propositions:

Proposition 1: If the *R-circle* intersects the *unit S-circle* on the complex *S-Plane* for a frequency ω then $|S| < 1$ and $|R| < 1$ can be satisfied simultaneously at this frequency.

Proposition 2: If the Ideal Disturbance Attenuation Problem is solvable then $\forall \omega$ the two circles $|S| < \sigma$ and $|R| < \varepsilon$ intersect on the complex *S-Plane.*

The above statement can of course be made mathematically precise by noting their geometric relationships but we shall not pursue this line of argument. We concentrate on the tradeoffs between S and R posed by the plants. From the above discussion it seems that arbitrarily small attenuation of S and R is not possible. It turns out that this becomes possible if the center of *R-circle* coincides with that of *unit S-circle.* The following proposition follows immediately:

Proposition 3: If $G = 1$, then arbitrarily small attenuation of both S and R is obtainable for a frequency ω.

In this case, it can be seen that there is no tradeoff between S and R. However $G = 1$ implies $G_{zw}G_{yu} = G_{zu}G_{yw}$ following equation (10). Assuming that G exists and is nonzero, then this is exactly the case defined to be "systems reducible to a feedback loop" in (Freudenberg et al. 2003). Both the case $G_{zw} = G_{zu}$ and $G_{yu} = G_{yw}$ and the case $G_{zw} = G_{yw}$ and $G_{yu} = G_{zu}$ represent two special classes of systems with particular sensor or actuator locations.

However the above results depend on the fact that G exists and is nonzero. To simplify the argument, consider the situation for a frequency ω, then G_{zw}, G_{zu}, G_{yw}, and G_{yu} are just complex numbers. Therefore that G does not exist implies that either G_{zu} or G_{yw} is zero while that G is zero implies that either G_{zw} or G_{yu} is zero. In the former case it is easy to see from equation (2) that in this case $T_{zw} = G_{zw}$ and hence $R = 1$ unless $G_{zw} = 0$ for which $R = 0$. Therefore there is no algebraic tradeoff between S and R. In the latter case, if $G_{yu} = 0$ then $S = 1$, while if $G_{zw} = 0$ then the exogenous input w cannot excite z directly. Thus for these "special cases", S and R acquire a certain degree of "independence". The interest, in our view, should be more in the implications of the control architecture to be selected, e.g. sensor/actuator locations relative to the controller input/exogenous input, other than the algebraic tradeoffs to be implied.

Thus it has by now been clear that the form and structure of the plants G_{zw}, G_{zu}, G_{yw}, and G_{yu} prescribe the location of the *R-circle* and they consequently determine the possible algebraic tradeoffs between S and R at a frequency ω. However as stated above, the problem of fundamental design limitation is how the control designs are *inherently constrained* such that the Ideal Disturbance Attenuation Problem is unsolvable. Proposition 1 provides a partial answer:

Proposition 4: If only the circle $|S| < \sigma$ does not intersect the mapping of the circle $|R| < \varepsilon$ on the complex *S-Plane* for a certain frequency ω, then the Ideal Disturbance Attenuation Problem is unsolvable.

[1] Following (Freudenberg et al. 2003, Doyle et al. 1992, Freudenberg and Looze 1988), limitations are termed as "algebraic" if they refer tradeoffs at the same frequency, otherwise as "analytic".

This has an important implication: for disturbance attenuation σ should be less than or equal to unity. However from the classical Bode's sensitivity integral theorem even the circle $|S| < \sigma \leq 1$ itself cannot be defined ultimately at some frequencies. Thus it is inferred immediately that the Ideal Disturbance Attenuation Problem is unsolvable for any design.

At this point, the investigation into fundamental design limitations can be pursued in two different directions: one is to modify the Ideal Disturbance Attenuation Problem into Proper Disturbance Attenuation Problem and analyze its solvability. This finally leads to a generalized Bode's integral relationship holding for R (see (Freudenberg et al. 2003)). However as remarked these integrals although explain the failure of some designs and can indeed be said to be "fundamental", they do not provide us with a specific design methodology. We thus shall proceed with an alternative route into the problem of fundamental design limitations, which also provides a design methodology. This is considered in the following section.

3. CONTROLLER WITH INTERNAL STABILITY GUARANTEE

Towards the design purpose, it is noted that the *tradeoff line* illustrated in Fig. 2 can be thought of as *design freedom* and a point $S(j\omega)$ chosen on it provides immediate information on the disturbance attenuation property. The controller can consequently be obtained from equation (3)

$$K(j\omega) = \frac{S(j\omega)-1}{S(j\omega)G_{yu}(j\omega)}. \quad (11)$$

For the broad-band case, depending on the control objective, S can be chosen frequency-by-frequency and results in a trajectory on the complex *S-Plane*. (See Fig. 3)

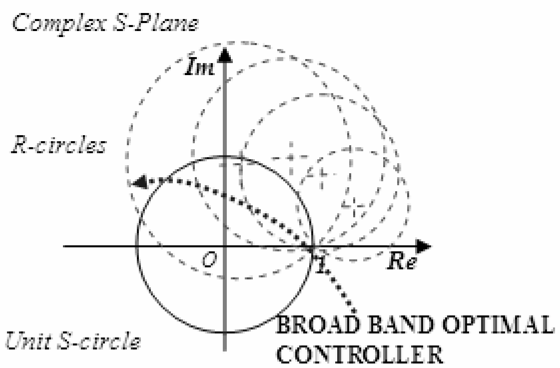

Fig. 3: Tradeoffs line is viewed as design freedom, then broadband optimal controller results in a trajectory on the complex *S-Plane*.

For the single frequency case (e.g. harmonic control) there is no problem of implementing the controller (11), except for the case where T_{jw} is required to be absolutely rejected since this will result in an infinite-gain controller but can nevertheless be achieved using a feed-forward structure. However for the broad band case the resulting controller has to be a stabilizing one or can provide internal stability. This is the fundamental design limitation posed by internal stability in this geometric framework.

It is indeed possible to obtain constraints on the design freedom S to give a stabilizing controller, e.g. by controller parameterization S can be parameterized by $Q \in \wp$ where $\wp$ is the set of all stable, proper and real-rational functions. However those constraints are easily obtained but very difficult to be utilized for design purposes in this geometric framework, if that is not impossible.

It turns out that if G_{yu} is assumed to be stable and minimum phase, which essentially means that $y-u$ can be collocated, then a particularly appealing geometric design methodology can be developed which provides internal stability guarantee. However any design limitations based on this assumption cannot be classified to be "fundamental" in the real sense. But this assumption can indeed be justified: as stated in the introduction this work concerns the case that performance variable is not measured. This case arises largely from the active noise and vibration control area, while it is in this area of engineering applications that collocated control is widely used even with deliberate deployment since it not only guarantees the asymptotic stability but also provides extreme robustness (Daley 2004, Preumont 2002). This is especially important for lightly damped flexible structures, which can be very difficulty to control otherwise (Inman 1990, MacMartin 1995). Even self-sensing piezoelectric actuators have been developed with the sole intention for collocated control (Dosch et al. 1992).

With this assumption, suppose $S = \alpha + 1$, then the following important result for strong stabilization can be proved:

Proposition 5: If G_{yu} is stable and minimum phase and α is a mapping of a stable function and in addition, $\mathrm{Re}(\alpha) > -0.5$ when $\mathrm{Im}(\alpha) = 0$, then the resulting controller (11) will provide internally stabilize the closed loop system and furthermore, the controller itself is stable.

Proof: From the definition $S = \alpha + 1$ and equation (11) the loop gain can be expressed as:

$$L = KG_{yu} = \frac{\alpha}{1+\alpha}. \quad (12)$$

This is equivalent to a closed loop system with loop gain α. Thus if α is a mapping of a stable function satisfying

$\text{Re}(\alpha) > -0.5$ when $\text{Im}(\alpha) = 0$, then L is stable and furthermore its Nyquist D-contour does not enclose the critical point in the complex α plane. These consequently imply that the original closed loop system is stable. Now if G_{yu} is stable and minimum phase then for a stable L, there cannot be any unstable pole-zero cancellations between G_{yu} and the controller K. This implies that the closed loop system is strongly stabilized by the controller. This completes the proof. ▮

The above result is important since S is the design freedom which shows the algebraic tradeoffs in disturbance attenuation property. The simple relationship $S = \alpha + 1$ means that α can also provide immediate information on the tradeoffs and can thus be "legitimately" viewed as the design freedom. Remarkably, if α is also such chosen as to satisfy the conditions in Proposition 5, then the resulting controller K expressed as equation (11) will provide strong stabilization. Thus in this geometric design framework, α becomes the ultimate design freedom. It is in this sense that it is claimed the geometric approach results in a new perspective towards control design—a design freedom is not necessarily the performance variable to be directly designed (compensator, for example) or minimized (sensitivity, for example) but a variable which can facilitate the overall design. It is also worthwhile pointing out that such a perspective is implicit in an analytic setup in (Newton et al. 1957) and in the field of process control where it is called *internal model control* (Morari and Zafiriou 1989).

Remark: It is very important to ensure the minimum-phase of G_{yu} since in practical engineering even collocated control can result in a non-minimum phase system. However, it can be shown that if G_{yu} is stable but non-minimum phase, e.g. existence of right-hand-plane zeros, then the *necessary and sufficient* condition for the closed loop system to be stabilizing is that α is a mapping of a stable function that also interpolates the unstable zeros of G_{yu}, although the controller itself is unstable. See appendix for the proof.

4. GEOMETRIC DESIGN PROCEDURES FOR HARMONIC AND BROADBAND CONTROL

The procedures of the geometric design methodology are summarized as follows.

4.1 Design Procedures for Harmonic Control

Step 1: From the system plant, determine the geometry of S and R;

Step 2: Depending on the design objective, choose appropriate $S(j\omega)$ point from the Tradeoff Line (Fig. 2);

Step 3: Implement the feedback controller $K(j\omega)$ as in (11).

4.2 Design Procedures for Broadband Control

Step 1: Choose control architecture so that $y-u$ is collocated (G_{yu} is thus stable);

Step 2: From the system plant, determine the geometry of S and R;

Step 3: Depending on the design objective, choose appropriate S trajectory (Fig. 3);

Step 4: Find a stable function $\alpha(s)$ such that it interpolates the optimal trajectory $\alpha = S - 1$ designed in step 3;

Step 5: Implement the feedback controller K as in (11).

It is noted that the crucial step (4) is not an easy task. But this can nevertheless be resolved by using a Nevanlinna-Pick interpolation approach that has been developed in (Wang and Daley, 2007). It is also noted that if G_{yu} is stable but non-minimum phase, then a stable interpolating function $\alpha(s)$ results in a stabilizing controller which is itself not stable; if G_{yu} is stable and minimum phase, then the stable interpolating function $\alpha(s)$ with enhanced condition of $\text{Re}(\alpha) > -0.5$ when $\text{Im}(\alpha) = 0$ results in a *strongly stabilizing* controller, which is highly desirable.

6. CONCLUSIONS

The fundamental design limitations for the case where the performance variable is not measured for feedback have been investigated and approached from a geometric viewpoint. The geometry is remarkable in that it illustrates clearly fundamental tradeoffs and hence provides intuition about solution existence and insight into control design. These properties of this geometric design methodology might not be retained otherwise. For example, in work (Freudenberg et al. 2003), it is stated that if $G \neq 1$ a tradeoff does exist and will be severe at any frequency where $|G|$ is either very large or very small. Such a statement is not very illuminating since if the centre of the *R-circle* locates in the left-hand half plane then S can still be made arbitrarily small even without enhancement of R, although $|G|$ can be very large. The same situation also holds for R: when $|G|$ is small but locates inside of the *S-circle*, then R can be made arbitrarily small without enhancement of S. These statements can be clearly demonstrated using the geometry of S and R.

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Appendix: PROOF OF THE FOLLOWING STATEMENT MADE IN THE REMARK OF SECTION 3

Proposition: If G_{yu} is stable, then the necessary and sufficient condition for the resulting controller (11) to be stabilizing is $\alpha \in \wp$ that also interpolates the unstable zeros of G_{yu}.

Proof: Assume $Q \in \wp$ and $\wp$ is the set of all stable and real-rational functions, then all stabilizing controller can be parameterized by

$$K = \frac{Q}{1 + G_{yu} Q}. \qquad (13)$$

Specifically we have

$$S = 1 + G_{yu} Q \qquad (14)$$

Meanwhile $S \equiv 1 + \alpha$ then

$$Q = \alpha G^{-1}{}_{yu} \qquad (15)$$

Hence $Q \in \wp$ is guaranteed by the condition on α stated above and $\wp$ is closed under multiplication. This completes the proof.