A NEW APPROACH TO INPUT-OUTPUT PAIRING ANALYSIS FOR UNCERTAIN MULTIVARIABLE PLANTS

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Abstract: In this paper, a new method to analyze the input-output pairing for uncertain multivariable plants is proposed. Here, Hankel Interaction Index Array is used to choose the appropriate input-output pair and a theorem will be presented to show the effect of additive uncertainties on input-output pairing of the system. In this theorem a new approach to compute the variation bound of Hankel Interaction Index Array elements due to additive uncertainties in state space framework is given to study the possible change in input-output pairing. Finally, two typical plants are employed to show the main points of the proposed methodology.

Keywords: Decentralized Control; Multivariable Plants; Input-Output Pairing; Additive Uncertainty.

1. INTRODUCTION

Decentralized controllers are used in many complex multivariable plants [1-6]. An appropriate input-output pairing prior to the commencement of the design is vital for desired closed-loop stability and performance. There are different approaches to input-output pairing and RGA is the first and the most widely used analytical tool for this problem [6,7]. However, in the face of unknown or uncertain multivariable plants, the input-output structure of the plant may endure fundamental changes, which will severely degrade the decentralized controller performance. The well-known input-output pairing techniques [7-10] are unable to analyze the effect of uncertainty on input-output pairing and only recently pairing methods are proposed for uncertain multivariable systems [11-15]. Moaveni and Khaki Sedigh in [20] analyze the effect of additive uncertainty in state space model on input-output pairing analysis based on the Hankel Interaction Index Array.

In this paper a new method, based on the proposed method in [20], to study the effect of uncertainty on input-output pairing is presented. Finally, a distillation column model is employed to show the effectiveness of the proposed methodology in the face of parameter uncertainties or variations.

2. INPUT-OUTPUT PAIRING USING HANKEL INTERACTION INDEX ARRAY

Gramian-Based [9] and Hankel Interaction Index Array [10] analyze the interaction dynamically using controllability and observability Gramian matrices for each elementary subsystem. In [10] a new interaction measure is proposed using $\overline{\Sigma}_H$, which is called Hankel Interaction Index Array. Hankel Interaction Index Array is defined as:

$$\overline{\Sigma}_H = [\overline{\Sigma}_H^{ij}] = [\overline{\sigma}(W_c^{ij} W_o^{ij})] \qquad (1)$$

where, W_c and W_o stand for controllability and observability Gramians, respectively. In other words, elements of $\overline{\Sigma}_H$, $[\overline{\Sigma}_H^{ij}]$, are the Hankel-norm of the elementary subsystem (A, b_{*i}, c_{j*}). It is well-known that Hankel singular values are invariant under different state space realizations [17], so, Moaveni and Khaki Sedigh in [20] proposed a new method to compute the Hankel Interaction Index Array as follows:

$$\overline{\Sigma}_H = [\max\{|\lambda(W_{co}^{ij})|\}] \qquad (2)$$

Where, W_{co}^{ij} is the cross-Gramian matrix related to elementary subsystem (A, b_{*i}, c_{j*}). Equation (2), considerably reduces the computational task of the methodology in comparison with the two Lyapunov equations to compute the controllability and observability Gramian matrices as in [9] or [10].

3. INPUT-OUTPUT PAIRING ANALYSIS IN THE PRESENCE OF UNCERTAINTY

To choose the appropriate input-output pair of linear multivariable systems in the presence of uncertainty, it is important to know the variations of Hankel Interaction Index Array elements resulting from the uncertainty [20]. In this paper, a general additive uncertainty in state space description of the plant is investigated and a cross-Gramian technique is used to compute the Hankel Interaction Index Array [20]. The uncertainty effects on the elements of this matrix are studied and variation bounds are given to avoid an input-output pairing change.

The following theorem provides the variation bounds of Hankel Interaction Index Array elements in the presence of additive uncertainty. Using these bounds, the input-output pairing of uncertain multivariable plants can be analyzed.

Theorem, Consider the linear multivariable plant realization (A,B,C) and the corresponding Hankel Interaction Index Array, $\overline{\Sigma}_H = [\overline{\sigma}(W_{cob}^{ij})] = [\max\{|\lambda(W_{co}^{ij})|\}]$. In the presence of additive uncertainty, we have:

$$\dot{X} = (A+\Delta A)X + (B+\Delta B)U = (A+\Delta A)X + \sum_{i=1}^{m}(b_{*i}+\Delta b_{*i})u_i \quad (3)$$
$$Y = (C+\Delta C)X \Rightarrow Y = [y_j] = [(c_{j*}+\Delta c_{j*})X]$$

where, b_{*i} and c_{j*} are i^{th} column of B and j^{th} row of C respectively. Also, ΔA is the additive uncertainty in matrix A, Δb_{*i} is the additive uncertainty in i^{th} column of B and Δc_{j*} is the additive uncertainty in j^{th} row of matrix C. It is assumed that the upper bounds of $\|\Delta A\|$, $\|\Delta b_{*i}\|$, $\|\Delta c_{j*}\|$ are known. Where, $\|\cdot\|$ denotes the 2-norm. If $\overline{\Sigma}_H^U = [\max\{|\lambda(W_{co}^{ij}+\Delta^{ij})|\}]$ is the corresponding Hankel Interaction Index Array for the uncertain multivariable plant, Δ^{ij} is the variation of matrix W_{co}^{ij} in the presence of uncertainty and $\overline{\sigma}(\Delta A) < 0.5\underline{\sigma}(A\oplus A^T)$ then the variation bound of $\max\{|\lambda(W_{co}^{ij}+\Delta^{ij})|\}$ is:

$$\max\left\{\frac{\left(\underline{\sigma}(b_{*i})-\overline{\sigma}(\Delta b_{*i})\right)\left(\underline{\sigma}(c_{j*})-\overline{\sigma}(\Delta c_{j*})\right)}{2\sqrt{n}\left(\overline{\sigma}(A)+\overline{\sigma}(\Delta A)\right)},0\right\} \le \quad (4)$$
$$\cdots \le \max\{|\lambda(W_{co}^{ij}+\Delta^{ij})|\} \le \frac{(\|b_{*i}\|+\|\Delta b_{*i}\|)(\|c_{j*}\|+\|\Delta c_{j*}\|)}{\underline{\sigma}(A\oplus A^T)-2\overline{\sigma}(\Delta A)}$$

where, "$\oplus$" operator shows the Kronecker sum. **Proof:** To compute the Hankel Interaction Index Array elements, $\overline{\Sigma}_H^{ij}$, matrix W_{co}^{ij} should be computed from the following equations:

$$W_{co}^{ij}A + AW_{co}^{ij} = -b_{*i}c_{j*} \quad (5)$$

In the presence of additive uncertainty, this equation is rewritten as:

$$(W_{co}^{ij}+\Delta^{ij})(A+\Delta A) + (A+\Delta A)(W_{co}^{ij}+\Delta^{ij}) = -(b_{*i}+\Delta b_{*i})(c_{j*}+\Delta c_{j*}) \quad (6)$$

and,

$$vec(W_{co}^{ij}+\Delta^{ij}) = -\left(I_n\otimes(A+\Delta A)+(A+\Delta A)^T\otimes I_n\right)^{-1}vec\left((b_{*i}+\Delta b_{*i})(c_{j*}+\Delta c_{j*})\right) \quad (7)$$

where, "$\otimes$" operator shows the Kronecker product and $\left(I_n\otimes(A+\Delta A)+(A+\Delta A)^T\otimes I_n\right)$ should be non-singular. By taking 2-norm:

$$\|vec(W_{co}^{ij}+\Delta^{ij})\| = \left\|\left(I_n\otimes(A+\Delta A)+(A+\Delta A)^T\otimes I_n\right)^{-1}vec\left((b_{*i}+\Delta b_{*i})(c_{j*}+\Delta c_{j*})\right)\right\| \quad (8)$$

using the norm properties and $\|vec(\cdot)\| = \|\cdot\|_F$ [18]:

$$\|W_{co}^{ij}+\Delta^{ij}\|_F = \|vec(W_{co}^{ij}+\Delta^{ij})\| \le \left\|\left(I_n\otimes(A+\Delta A)+(A+\Delta A)^T\otimes I_n\right)^{-1}\right\|\left\|(b_{*i}+\Delta b_{*i})(c_{j*}+\Delta c_{j*})\right\|_F \quad (9)$$

hence:

$$\|W_{co}^{ij}+\Delta^{ij}\| \le \left\|\left(I_n\otimes(A+\Delta A)+(A+\Delta A)^T\otimes I_n\right)^{-1}\right\| \times \|b_{*i}+\Delta b_{*i}\|_F\|c_{j*}+\Delta c_{j*}\|_F \quad (10)$$

using the singular value inequalities [19]:

$$\|W_{co}^{ij}+\Delta^{ij}\| \le \frac{\|b_{*i}+\Delta b_{*i}\|_F\|c_{j*}+\Delta c_{j*}\|_F}{\underline{\sigma}(I_n\otimes(A+\Delta A)+(A+\Delta A)^T\otimes I_n)} = \frac{\|b_{*i}+\Delta b_{*i}\|_F\|c_{j*}+\Delta c_{j*}\|_F}{\underline{\sigma}\left((A\oplus A^T)+(\Delta A\oplus\Delta A^T)\right)} \quad (11)$$

where, "$\oplus$" operator shows the Kronecker sum and $A\oplus A^T = I_n\otimes A + A^T\otimes I_n$ [21].

It is important to know that $\|\cdot\|_F$ is equal $\|\cdot\|_2$ for vectors, So:

$$\|W_{co}^{ij}+\Delta^{ij}\| \le \frac{\|b_{*i}+\Delta b_{*i}\|\|c_{j*}+\Delta c_{j*}\|}{\underline{\sigma}\left((A\oplus A^T)+(\Delta A\oplus\Delta A^T)\right)} \quad (12)$$

Where, if $\overline{\sigma}(\Delta A\oplus\Delta A^T) < \underline{\sigma}(A\oplus A^T)$, then equation (12) can be rewritten as:

$$\|W_{co}^{ij}+\Delta^{ij}\| \le \frac{\|b_{*i}+\Delta b_{*i}\|\|c_{j*}+\Delta c_{j*}\|}{\underline{\sigma}(A\oplus A^T)-\overline{\sigma}(\Delta A\oplus\Delta A^T)} \quad (13)$$

Also, we know that $\overline{\sigma}(\Delta A\oplus\Delta A^T) \le 2\overline{\sigma}(\Delta A)$, So if $2\overline{\sigma}(\Delta A) < \underline{\sigma}(A\oplus A^T)$ then:

$$\|W_{co}^{ij}+\Delta^{ij}\| \le \frac{(\|b_{*i}\|+\|\Delta b_{*i}\|)(\|c_{j*}\|+\|\Delta c_{j*}\|)}{\underline{\sigma}(A\oplus A^T)-2\overline{\sigma}(\Delta A)} \quad (14)$$

Therefore, we find the upper bound of the element variation bound of Hankel Interaction Index Array due to the unstructured uncertainty in plant, but we should know the lower bound of this variation bound. So, we can rewrite equation (14) as follows:

$$\frac{\left\|vec\left((b_{*i}+\Delta b_{*i})(c_{j*}+\Delta c_{j*})\right)\right\|}{\overline{\sigma}(I_n\otimes(A+\Delta A)+(A+\Delta A)^T\otimes I_n)} \le \|vec(W_{co}^{ij}+\Delta^{ij})\| \le \|W_{co}^{ij}+\Delta^{ij}\|_F \quad (15)$$

Hence:

$$\frac{\left\|(b_{*i}+\Delta b_{*i})(c_{j*}+\Delta c_{j*})\right\|_F}{\overline{\sigma}(A\oplus A^T)+\overline{\sigma}(\Delta A+\Delta A^T)} \tag{16}$$
$$\leq \left\|vec(W_{co}^{ij}+\Delta^{ij})\right\| \leq \sqrt{n}\left\|W_{co}^{ij}+\Delta^{ij}\right\|$$

Using norm inequalities $\overline{\sigma}(\Delta A\oplus \Delta A^T)\leq 2\overline{\sigma}(\Delta A)$ and $\overline{\sigma}(A\oplus A^T)\leq 2\overline{\sigma}(A)$:

$$\max\left\{\frac{\left(\underline{\sigma}(b_{*i})-\overline{\sigma}(\Delta b_{*i})\right)\left(\underline{\sigma}(c_{j*})-\overline{\sigma}(\Delta c_{j*})\right)}{2\left(\overline{\sigma}(A)+\overline{\sigma}(\Delta A)\right)},0\right\} \tag{17}$$
$$\leq \frac{\left\|(b_{*i}+\Delta b_{*i})(c_{j*}+\Delta c_{j*})\right\|_F}{\overline{\sigma}(A\oplus A^T)+\overline{\sigma}(\Delta A\oplus \Delta A^T)} \leq \sqrt{n}\left\|W_{co}^{ij}+\Delta^{ij})\right\|$$

Hence, using (40) and (42):

$$\max\left\{\frac{\left(\underline{\sigma}(b_{*i})-\overline{\sigma}(\Delta b_{*i})\right)\left(\underline{\sigma}(c_{j*})-\overline{\sigma}(\Delta c_{j*})\right)}{2\sqrt{n}\left(\overline{\sigma}(A)+\overline{\sigma}(\Delta A)\right)},0\right\} \tag{18}$$
$$\leq \left\|W_{co}^{ij}+\Delta^{ij}\right\| \leq \frac{(\|b_{*i}\|+\|\Delta b_{*i}\|)(\|c_{j*}\|+\|\Delta c_{j*}\|)}{\underline{\sigma}(A\oplus A^T)-2\overline{\sigma}(\Delta A)}$$

So, the variation bound of the elements of Hankel Interaction Index Array due to the unstructured uncertainty in plant is as follows:

$$\max\left\{\frac{\left(\underline{\sigma}(b_{*i})-\overline{\sigma}(\Delta b_{*i})\right)\left(\underline{\sigma}(c_{j*})-\overline{\sigma}(\Delta c_{j*})\right)}{2\sqrt{n}\left(\overline{\sigma}(A)+\overline{\sigma}(\Delta A)\right)},0\right\} \tag{19}$$
$$\leq \max\left\{\left|\lambda(W_{co}^{ij}+\Delta^{ij})\right|\right\} \leq \frac{(\|b_{*i}\|+\|\Delta b_{*i}\|)(\|c_{j*}\|+\|\Delta c_{j*}\|)}{\underline{\sigma}(A\oplus A^T)-2\overline{\sigma}(\Delta A)}$$

■

The above theorem provides an upper bound for the element variations in Hankel Interaction Index Array due to the unstructured uncertainty in the plant.

An algorithm is now proposed that using the above theorem can show the possible changes in input-output pairing resulting from the model uncertainty.

Algorithm:

Step 1: Calculate the variation bounds of Hankel Interaction Index Array elements using (4).

Step 2: Input-output pairing analysis by variation bounds and their overlaps:

- ♦ If there is no overlap between variation bounds of the same row and the same column in Hankel Interaction Index Array, the nominal input-output pairing remains valid for all parameter variations.
- ♦ If there is an overlap between variation bounds of the same row or the same column in Hankel Interaction Index Array, the nominal input-output pairing may change due to parameter variations.

4. SIMULATION RESULTS

In this section, simulation results are used to illustrate the effectiveness of the approach to input-output pairing of uncertain linear multivariable plants.

Example 1:

Consider the well-known Wood and Berry binary distillation column process as

$$\begin{bmatrix} X_D(s) \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix} \tag{20}$$

where, X_D and X_B are the overhead and bottom compositions of methanol, respectively, R is the reflux flow rate, and S is the steam flow rate to the reboiler [17].

Using the above method to choose the appropriate input-output pairing, Hankel Interaction Index Array for the nominal model is as follows:

$$\overline{\Sigma}_H = \begin{bmatrix} 6.7527 & 10.5759 \\ 4.4813 & 11.2746 \end{bmatrix} \tag{21}$$

where, a first order Pade approximation to realize the time delays are used and (u_1-y_1, u_2-y_2) is the appropriate input-output pair similar to the RGA analysis in [12].

Two cases are considered to analyze the effect of uncertainty on the input-output pairing analysis. We consider the matrix transfer function of Wood-Berry distillation column as:

$$\begin{bmatrix} X_D(s) \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{k_{11}e^{-1s}}{\tau_{11}s+1} & \dfrac{k_{12}e^{-3s}}{\tau_{12}s+1} \\ \dfrac{k_{21}e^{-7s}}{\tau_{21}s+1} & \dfrac{k_{22}e^{-3s}}{\tau_{22}s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix} \tag{22}$$

In 1st case, the parameter variations look as:

$$\begin{aligned} &k_{11}\in[12.75,\ \ 12.85],\ \ k_{12}\in[-18.92,\ \ -18.88],\\ &k_{21}\in[6.55,\ \ 6.65],\ \ k_{22}\in[-19.45,\ \ -19.35]\\ &\tau_{11}\in[16.67,\ \ 16.73],\ \ \tau_{12}\in[20.98,\ \ 21.02],\\ &\tau_{21}\in[10.8,\ \ 11.0],\ \ \tau_{22}\in[14.37,\ \ 14.44] \end{aligned} \tag{23}$$

Therefore, the variation bounds of the elements of uncertain Hankel Interaction Index Array are:

$$\overline{\Sigma}_H^U = \begin{bmatrix} 5.2325\leq\overline{\sigma}(W_{co}^{11}+\Delta^{11})\leq 8.7390 & 9.5256\leq\overline{\sigma}(W_{co}^{12}+\Delta^{12})\leq 10.7962 \\ 1.1674\leq\overline{\sigma}(W_{co}^{21}+\Delta^{21})\leq 7.2638 & 10.8496\leq\overline{\sigma}(W_{co}^{22}+\Delta^{22})\leq 12.8127 \end{bmatrix} \tag{24}$$

So, for assumed uncertainty (u_1-y_1, u_2-y_2) remains the appropriate pair, because the lower bound of $\overline{\Sigma}_H^{U22}$ is greater than the upper bounds of $\overline{\Sigma}_H^{U12}$ and $\overline{\Sigma}_H^{U21}$.

In 2nd case we consider the parameter variations as:

$$\begin{aligned} &k_{11}\in[12.7,\ \ 12.9],\ \ k_{12}\in[-18.95,\ \ -18.85],\\ &k_{21}\in[6.5,\ \ 6.7],\ \ k_{22}\in[-19.55,\ \ -19.25]\\ &\tau_{11}\in[16.6,\ \ 16.8],\ \ \tau_{12}\in[20.9,\ \ 21.1],\\ &\tau_{21}\in[10.7,\ \ 11.1],\ \ \tau_{22}\in[14.3,\ \ 14.5] \end{aligned} \tag{25}$$

Hence, the variation bounds of the elements of the uncertain Hankel Interaction Index Array are:

$$\overline{\Sigma}_H^U = \begin{bmatrix} 4.3482\leq\overline{\sigma}(W_{co}^{11}+\Delta^{11})\leq 9.9405 & 8.5256\leq\overline{\sigma}(W_{co}^{12}+\Delta^{12})\leq 12.2859 \\ 0\leq\overline{\sigma}(W_{co}^{21}+\Delta^{21})\leq 9.8429 & 7.8740\leq\overline{\sigma}(W_{co}^{22}+\Delta^{22})\leq 14.2841 \end{bmatrix} \tag{26}$$

It is readily seen that the upper bounds of $\overline{\Sigma}_H^{U12}$ and $\overline{\Sigma}_H^{U21}$ can be greater than the lower bounds of $\overline{\Sigma}_H^{U22}$ and $\overline{\Sigma}_H^{U11}$, respectively. Hence, there is a possible input-output pairing change in the Wood and Berry distillation column process.

5. CONCLUSION

In this paper, an approach based on the cross-Gramian matrix is introduced to compute the Hankel Interaction Index Array of the plant. The proposed approach leads to a theorem to analyze the input-output pairing for uncertain multivariable plants. This facilitates input-output pairing analysis in the face of additive uncertainties in state space models. Simulation results are used to show the effectiveness of the proposed methodology.

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