Stabilizing Systems with Aperiodic Sample-and-Hold Devices: State Feedback Case

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Abstract: Motivated by the widespread use of networked and/or embedded control systems, an algorithm for stabilizing sampled-data feedback control systems with uncertainly time-varying sampling intervals is proposed, where it is assumed that the sampled state is available for feedback. The algorithm is an extension of that for stability analysis in the authors' previous study, and is based on the robustness against the variation of sampling intervals derived by the small-gain condition. The validity of the algorithm is demonstrated by numerical examples.

Keywords: networked control systems, sampled-data systems, quadratic stability, matrix exponential

1. INTRODUCTION

The sampled-data control theory (See Chen and Francis [1995] and references therein) has been well-developed in the last two decades. One of the crucial properties in the development of the sampled-data control theory is the periodicity of the closed-loop systems which comes from the periodic sampling.

We, however, recently encounter applications where the periodic sampling is impossible. For example, resources for measurement and control are restricted in networked and/or embedded control systems (See Hristu-Varsakelis and Levine [2005], Hespanha et al. [2007] and references therein), and hence the sampling operation results to be aperiodic and uncertain. Hence it is obvious that robustness against variation of sampling intervals are quite important for such applications. One can find pioneering work in the literature Walsh et al. [1999], Zhang et al. [2001], Zhang and Branicky [2001].

The so-called input delay approach Fridman et al. [2004] was proposed to treat the systems with aperiodic sampling. The basic idea of the approach is modeling the aperiodic sample-and-hold operations by a time-varying uncertain time delay at control input, and hence one can apply methodologies developed for delay systems to the aperiodic sampled-data systems. One can find applications of the input delay approach to several analysis and synthesis problems Fridman et al. [2004, 2005], Naghshtabrizi and Hespanha [2005], Suplin et al. [2007]. Moreover this approach has inspired the discussion of the problem from the viewpoints of hybrid systems Naghshtabrizi et al. [2006] and robust control Mirkin [2007], Fujioka [2007]. It would be worth mentioning that most of existing results verify stability by showing the existence of a continuous-time Lyapunov function.

On the other hand, Fujioka [2008] recently proposed an algorithm to check the stability of the aperiodic sampled-data systems, where a discrete-time approach is taken and the stability is verified by showing the existence of a discrete-time quadratic Lyapunov function. A similar approach was taken in Zhang and Branicky [2001] where a quadratic Lyapunov function is searched by a randomized algorithm. The algorithm in Zhang and Branicky [2001], however, checks the quadratic stability only on the finite grid between bounds of sampling intervals. In contrast the algorithm in Fujioka [2008] checks the quadratic stability for all sampling intervals uncertainly varying between given lower and upper bounds, by exploiting the robustness against the variation of sampling intervals based on the small-gain condition. Moreover, it is demonstrated that the algorithm in Fujioka [2008] is less conservative than existing methods for some examples.

The purpose of this paper is to apply the method in Fujioka [2008] to a synthesis problem. In particular, we will provide a synthesis algorithm of constant feedback gain matrix which exponentially stabilizes the resulting system, under the condition that the sampled state is available for feedback although the sampling intervals are uncertainly time-varying.

This paper is organized as follows: The problem is formulated in Section 2. Section 3 provides a synthesis algorithm to exponentially stabilize resulting systems. The validity of the algorithm is demonstrated in Section 4.

2. PROBLEM FORMULATION

Let the following state-space system be given

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where x and u respectively denote the state and the input taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$. A and B are matrices of compatible dimensions.

We consider the following scenario of the feedback control of (1):

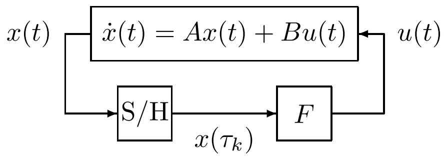

Fig. 1. Feedback control with aperiodic sample-and-hold circuits

- We can measure the state of (1) when $t = \tau_k$ ($k = 0, 1, \ldots$) where $\{\tau_k\}$ is an uncertain set of discrete time instances satisfying $\tau_0 = 0$ and
$$0 < h_\ell \le \tau_{k+1} - \tau_k \le h_u < \infty \qquad (2)$$
for given h_ℓ and h_u.
- The control input u is determined by the sampled state $x(\tau_k)$ and a constant feedback gain $F \in \mathbb{R}^{m\times n}$, which is the design parameter in this paper, with the zero-th order hold synchronizing the sampling:
$$u(t) = Fx(\tau_k), \quad {}^\forall t \in [\tau_k,\, \tau_{k+1}). \qquad (3)$$

Remark 1. The condition $h_\ell > 0$ implies
$$\lim_{k\to\infty} \tau_k = \infty, \qquad (4)$$
i.e., it is assumed that the Zeno phenomena does not happen.

The resulting feedback system composed of (1) and (3), denoted by T, is given by
$$\dot{x}(t) = Ax(t) + BFx(\tau_k), \quad {}^\forall t \in [\tau_k,\, \tau_{k+1}). \qquad (5)$$
See also Fig. 1. Applications of this scenario can be found in networked and/or embedded control systems Hristu-Varsakelis and Levine [2005], Hespanha et al. [2007], where resources for measurement and control are restricted.

The purpose of this paper is to provide a synthesis algorithm of F for stabilizing T. If τ_k's satisfy
$$\tau_{k+1} - \tau_k = \tilde{h}$$
for some $\tilde{h} \in [h_\ell,\, h_u]$, the resulting feedback control system T is periodic. This special scenario is the one well-studied in the so-called sampled-data control theory Chen and Francis [1995]. Indeed the stability can be easily verified by checking the spectral radius of $\Phi(\tilde{h})$ in the special scenario, where
$$\Phi_F(h) := \Phi(h) + \Gamma(h)F, \qquad (6)$$
$$\Phi(h) := \mathrm{e}^{Ah}, \quad \Gamma(h) := \int_0^h \mathrm{e}^{A(h-\eta)} B\,\mathrm{d}\eta,$$
and finding a stabilizing F is routine. It is, however, readily to see that our general scenario is much more complicated, because of the uncertainly time-varying nature.

In this paper we will develop a stabilization algorithm based on the following lemma Zhang and Branicky [2001], Hespanha et al. [2007]:

Lemma 1. T is exponentially stable if there exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying
$$(\Phi_F(h))^* P \Phi_F(h) - P < 0 \qquad (7)$$
for all $h \in [h_\ell,\, h_u]$, where $\Phi(\cdot)$ is defined in (6).

Note that Lemma 1 is based on the quadratic stability of the accompanying discrete-time system T_d defined by
$$\xi[k+1] = \Phi_F(\tau_{k+1} - \tau_k)\xi[k]$$
with the discrete-time Lyapunov function
$$V(\xi[k]) := \xi^*[k]P\xi[k]$$
where $\xi[k] := x(\tau_k)$.

Note also that it is hard to find a matrix P in Lemma 1 since the inequality must hold for all values in $[h_\ell,\, h_u]$. In the previous study Fujioka [2008] we have developed a stability analysis algorithm based on Lemma 1 for a given F. In this paper we must find P and F simultaneously.

3. MAIN RESULTS

In the previous study of Fujioka [2008] we have developed an algorithm to construct a finite grid:
$$\mathcal{G} = \{h_1,\, h_2,\, \ldots,\, h_N\} \subset [h_\ell,\, h_u]$$
so that the existence of $P > 0$ satisfying (7) for all $h \in \mathcal{G}$ guarantees the existence of $P > 0$ satisfying (7) for all $h \in [h_\ell,\, h_u]$. In other words, we have provided an estimate of the robustness of systems with uniform sampling interval against the perturbation caused by the variation of sampling interval. In this section this idea will be applied to the synthesis problem.

3.1 Robust Stabilization against Variation of Sampling Intervals

In order to discuss the robustness against the variation of sampling interval, we consider the following manipulation of Φ_F: Fix $h_0 \in (h_\ell,\, h_u)$ and then one can define θ_k so that
$$\tau_{k+1} - \tau_k = h_0 + \theta_k.$$
One has the following property found in Fujioka [2008], which is simple but plays a key role in this paper:

Proposition 2. The function $\Phi(\cdot)$ defined in (6) satisfies
$$\Phi_F(\tau_{k+1} - \tau_k) = \Phi_F(h_0) + \Delta(\theta_k)\Psi_F(h_0) \qquad (8)$$
where
$$\Psi_F(h) := \Psi(h) + \Upsilon(h)F, \qquad (9)$$
$$\Psi(h) := A\Phi(h), \quad \Upsilon(h) := A\Gamma(h) + B,$$
$$\Delta(\theta) := \int_0^\theta \mathrm{e}^{A\eta}\,\mathrm{d}\eta.$$

Proof. By definition
$$\Phi_F(\tau_{k+1} - \tau_k) = \Phi(h_0 + \theta_k) + \Gamma(h_0 + \theta_k)F.$$
The first term can be transformed to
$$\Phi(h_0 + \theta_k) = \mathrm{e}^{A\theta_k}\Phi(h_0) = (I + \Delta(\theta_k)A)\,\Phi(h_0).$$
While for the second term we have the following:
$$\begin{aligned}
&\Gamma(h_0 + \theta_k)\\
&= \int_0^{h_0} \mathrm{e}^{A(h_0+\theta_k-\eta)} B\,\mathrm{d}\eta + \int_{h_0}^{h_0+\theta_k} \mathrm{e}^{A(h_0+\theta_k-\eta)} B\,\mathrm{d}\eta\\
&= \mathrm{e}^{A\theta_k}\Gamma(h_0) + \Delta(\theta_k)B\\
&= (I + \Delta(\theta_k)A)\Gamma(h_0) + \Delta(\theta_k)B.
\end{aligned}$$
Then it is straightforward to derive (8) by substituting the above results.

Now one can regard T_d as a feedback connection of an LTI discrete-time system $\mathcal{F}_\ell\,(\Sigma,\, F)$:
$$\Sigma : \begin{bmatrix} \xi[k+1] \\ z[k] \\ \xi[k] \end{bmatrix} = \begin{bmatrix} \Phi(h_0) & I & \Gamma(h_0) \\ \Psi(h_0) & 0 & \Upsilon(h_0) \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi[k] \\ w[k] \\ v[k] \end{bmatrix}.$$

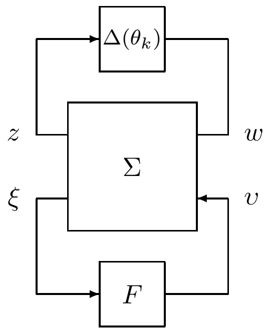

Fig. 2. Alternative representation of T_d

and a time-varying matrix $\Delta(\theta_k)$. See Fig. 2. Thus we obtain the following lemma as a simple application of the small-gain theorem [1]:

Lemma 3. Let an interval $\mathcal{H} \subseteq (0, \infty)$ be given. There exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying (7) for all $h \in \mathcal{H}$ if $\rho(\Phi(h_0)) < 1$ and

$$\gamma \|\Delta(\theta)\| \leq 1 \tag{10}$$

for all $\theta \in \mathcal{H} - h_0$, where γ is an upper bound of $\|\mathcal{F}_\ell(\Sigma, F)\|_\infty$:

$$\gamma > \|\mathcal{F}_\ell(\Sigma, F)\|_\infty . \tag{11}$$

Since minimization of γ in (11) is routine, one can stabilize T based on (10) by bounding $\|\Delta(\theta)\|$. For the purpose we invoke the following property in, e.g., Van Loan [1977]:

Lemma 4. For given $A \in \mathbb{R}^{n\times n}$ and $t \geq 0$ one has

$$\left\|e^{At}\right\| \leq e^{\mu(A)t} \tag{12}$$

where $\mu(A)$ denotes the log norm of A:

$$\mu(A) = \lambda_{\max}\left(\frac{A + A^*}{2}\right).$$

Remark 2. One can continue the following discussion by replacing the bound in (12) by other bounds found in, e.g., Van Loan [1977], Kågström [1977]. Indeed a similar study with a bound based on the Shur decomposition of A can be found in Suh [2008].

Hence we have the following basic robustness results, which is an alternative representation of Theorem 1 in Fujioka [2008]:

Theorem 5. Let $h_0 > 0$, F, and $\gamma > 0$ be given so that $\rho(\Phi_F(h_0)) < 1$ and (11). There exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying (7) for all $h \in \mathcal{H}(h_0, \gamma)$, where

$$\mathcal{H}(h, \gamma) := [h_L, h_U] \cap (0, \infty), \tag{13}$$

h_L and h_U are given as follows:

L1) If $\mu(-A) = 0$, $h_L = h - \gamma^{-1}$,
L2) else if $\mu(-A) \leq -\gamma$, $h_L = -\infty$,
L3) else

$$h_L = h - \frac{1}{\mu(-A)} \log\left(1 + \gamma^{-1}\mu(-A)\right).$$

U1) If $\mu(A) = 0$, $h_U = h + \gamma^{-1}$,
U2) else if $\mu(A) \leq -\gamma$, $h_U = \infty$,
U3) else

$$h_U = h + \frac{1}{\mu(A)} \log\left(1 + \gamma^{-1}\mu(A)\right).$$

Proof. See Appendix.

3.2 Algorithm for State Feedback Synthesis

Theorem 5 provides a robustness condition for T based on the property of the nominal system determined by the fixed sampling period h_0. Hence it is straightforward to apply Theorem 5 to the state feedback synthesis problem, i.e., the stabilization problem of T is cast into an $\mathbf{H}_\infty$ control problem for Σ with an appropriate $h_0 > 0$, and the $\mathbf{H}_\infty$ control gain F stabilizes T provided that $[h_\ell, h_u] \subseteq \mathcal{H}(h_0, \gamma)$ is satisfied. This direct use of Theorem 5, however, can be conservative in the sense that there might not exist $h_0 > 0$ and F such that $[h_\ell, h_u] \subseteq \mathcal{H}(h_0, \gamma)$ even though there exists a matrix P satisfying (7) for all $h \in [h_\ell, h_u]$, mainly because of the small-gain type modeling of $\Delta(\theta_k)$.

In order to reduce the conservatism we introduce the method similar to the one in the multi-objective problem to obtain the following theorem:

Theorem 6. Let $h_i > 0$ ($i = 1, 2, \ldots, N$) be given. If there exist a matrix $0 < X = X^* \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{m\times n}$, and $\alpha_i > 0$ ($i = 1, 2, \ldots, N$) satisfying the following N linear matrix inequalities

$$\begin{bmatrix} X & 0 & L_1(h_i, X, W) & I \\ 0 & \alpha_i I & L_2(h_i, X, W) & 0 \\ L_1^*(h_i, X, W) & L_2^*(h_i, X, W) & X & 0 \\ I & 0 & 0 & I \end{bmatrix} > 0 \tag{14}$$

then (7) is satisfied with $P = X^{-1}$, $F = WX^{-1}$ for all

$$h \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \sqrt{\alpha_i})$$

where $\Phi(\cdot)$, $\Gamma(\cdot)$, $\Psi(\cdot)$, $\Upsilon(\cdot)$, $\mathcal{H}(\cdot, \cdot)$ are defined in (6), (9), and (13), respectively, and

$$L_1(h_i, X, W) := \Phi(h_i)X + \Gamma(h_i)W,$$
$$L_2(h_i, X, W) := \Psi(h_i)X + \Upsilon(h_i)W.$$

Proof. Consider the case $i = 1$. The condition (14) with $i = 1$ is an equivalent representation of

$$\left\|\Psi_F(h_1)(zI - \Phi_F(h_1))^{-1}\right\|_\infty < \sqrt{\alpha_i}.$$

Hence, by invoking Theorem 5, there exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying (7) for all $h \in \mathcal{H}(h_1, \sqrt{\alpha_1})$. Moreover we can verify that one of such P is given by X^{-1} from the standard procedure. With similar discussion, we can conclude that there exists a matrix $0 < P = P^* = X^{-1} \in \mathbb{R}^{n\times n}$ satisfying (7) for all $h \in \mathcal{H}(h_i, \sqrt{\alpha_i})$, $i = 2, \ldots, N$. This concludes the proof.

Remark 3. Since F must be shared by all the h_i's, one cannot apply the so-called variable elimination method in the reduction to LMIs. This fact makes the numerical condition of the synthesis problem harder in compare to that of the analysis problem.

Once we find a pair of matrices $P > 0$ and F satisfying (7) on a grid by any methods, we can verify the robustness by

[1] Readers are referred to, e.g., Khargonekar et al. [1990] on the relationship between the quadratic stability and the small-gain condition.

invoking Theorem 6. In this paper we propose the following concrete algorithm for stabilization which generates a grid effectively based on Theorem 6. We denote the number of elements in a finite set $\mathcal{G}$ by $\#(\mathcal{G})$.

Algorithm 1. Given $0 < h_\ell < h_u < \infty$, and a large positive integer N_0.

0. Initialization: $\mathcal{G} \leftarrow \{(h_\ell + h_u)/2\}$

1. If there exists an $h \in \mathcal{G}$ such that $(\Phi(h_0), \Gamma(h_0))$ is not stabilizable, there is no F which stabilizes T. Stop.

2. If $\#(\mathcal{G}) \geq N_0$, stop without obtaining a stabilizing F.

3. Minimize

$$\sum_{i=1}^{\#(\mathcal{G})} \beta_i$$

subject to

$$\begin{bmatrix} X & 0 & L_1(h_i, X, W) & I \\ 0 & \beta_i I & L_2(h_i, X, W) & 0 \\ L_1^*(h_i, X, W) & L_2^*(h_i, X, W) & X & 0 \\ I & 0 & 0 & I \end{bmatrix} > 0$$

for all i's over $0 < X = X^* \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{m\times n}$, and $\beta_i \geq 0$, where h_i is the i-th smallest element in $\mathcal{G}$.

4. If

$$[h_\ell, h_u] \subseteq \bigcup_{i=1}^{\#(\mathcal{G})} \mathcal{H}(h_i, \sqrt{\alpha_i}),$$

T is exponentially stabilized by $F := WX^{-1}$. Stop. Here

$$\alpha_i := \lambda_{\max}(R_i - S_i^*(Q_i - X_i)^{-1}S_i) + \varepsilon$$

where ε is a small positive number and

$$\begin{bmatrix} Q_i & S_i \\ S_i^* & R_i \end{bmatrix} := \begin{bmatrix} \Phi_F(h_i) & I \\ \Psi_F(h_i) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_F(h_i) & I \\ \Psi_F(h_i) & 0 \end{bmatrix}^*.$$

5. Update $\mathcal{G}$ by

$$\mathcal{G} \leftarrow \mathcal{G} \cup \{(L_j + U_j)/2\}$$

for all j where L_j and U_j are determined so that

$$\bigcup_{j=1}^{M} (L_j, U_j) = (h_\ell, h_u) \setminus \bigcup_{i=1}^{\#(\mathcal{G})} \mathcal{H}(h_i, \sqrt{\alpha_i}),$$

$$L_1 < U_1 < L_2 < U_2 < \cdots < L_M < U_M$$

are satisfied. Go to Step 1.

We have some remarks for Algorithm 1: Step 2 is introduced to avoid numerical issues which could happen when $\#(\mathcal{G})$ is too large. The performance of the algorithm can be tuned by modifying the objective function in Step 3. Note that α_i satisfies (14) with X determined in Step 3 and $\alpha_i \leq \beta_i$ with sufficiently small ε. The integer M in Step 4 is $\#(\mathcal{G}) + 1$ at most.

4. NUMERICAL EXAMPLES

In this section we demonstrate the validity of the proposed method for stability synthesis.

Let us consider the following problem parameters Zhang and Branicky [2001]:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

The proposed algorithm provides a stabilizing gain

$$F = -[0.94 \ 6.38]$$

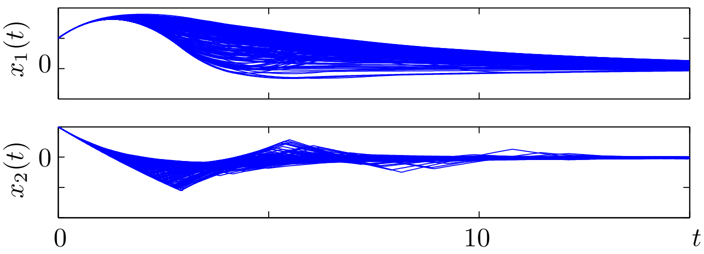

Fig. 3. Initial value responses

after 6 times loop with $\#(\mathcal{G}) = 25$ for $h_\ell = 0.01$ and $h_u = 2.93$, where we have used Robust Control Toolbox on MATLAB [2] as an LMI solver.

Fig. 3 denotes the 100 initial value responses with $x(0) = [1 \ 1]^*$ where $\{\tau_k\}$'s are randomly generated for $h_\ell = 0.01$ and $h_u = 2.93$.

The analysis algorithm in Fujioka [2008] proves that T is exponentially stable for $h_\ell = 0.01$ and $h_u = 3.12$ with the above F. On the contrary the proposed algorithm with Robust Control Toolbox on MATLAB fails to obtain a stabilizing F for $h_\ell = 0.01$ and $h_u = 2.94$, since $\#(\mathcal{G}) \geq N_0$ or the LMI solver fails. This phenomena depends on the performance of LMI solvers. In fact we can obtain stabilizing F for $h_\ell = 0.01$ and $h_u = 4.44$ with SDPT3 in Tutuncu et al. [2003].

5. EXTENSIONS FOR CONSERVATISM REDUCTION

The proposed algorithm chops the given interval $[h_\ell, h_u]$ into pieces to obtain a stabilizing gain F by using Theorem 5. Although it helps to cope with a large range of the sampling interval in spite of the conservatism in Theorem 5, it is obvious that the performance of the algorithm is improved if one can reduce the conservatism in Theorem 5. There are several directions for the purpose. In this section we suggest and discuss some of them with numerical evaluation.

A straightforward way is to replace the bound of the maximal singular value of matrix exponential in (12) by other bounds found in, e.g., Van Loan [1977], Kågström [1977]. Since the performance of the bound depends on the matrix taken the exponential Van Loan [1977], Kågström [1977], which is the 'A'-matrix of the plant in our problem, it might be practical to use bounds as many as possible if computational time is allowed.

Another way to reduce the conservatism is to replace the small gain condition (10) by a general quadratic condition in, e.g., Megretski and Rantzer [1997], Iwasaki and Hara [1998], Scherer [2000]. For the generalization it is required to find a matrix $\Pi = \Pi^* \in \mathbb{R}^{(n+m)\times(n+m)}$ satisfying

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0$$

for all $\Delta \in \{\Delta(\theta), \theta \in [h_\ell - h_0, h_u - h_0]\}$. One such Π is given by

$$\Pi = \begin{bmatrix} e^{-A^*\alpha}e^{-A\alpha} & 0 \\ 0 & -\gamma^2 e^{-A^*\alpha}e^{-A\alpha} \end{bmatrix}$$

[2] http://www.mathworks.com/products/robust/

for all $\alpha \in \mathbb{R}$ and $\gamma \leq 1/\|\Delta\|$, noting that $\Delta(\theta)$ and $e^{A\alpha}$ commute. Note that this Π is related to the scaled small gain condition and one can reduce the conservatism in Theorem 5 by replacing $\|\mathcal{F}_\ell(\Sigma, F)\|_\infty$ by $\left\|e^{-A\alpha}\mathcal{F}_\ell(\Sigma, F)\, e^{A\alpha}\right\|_\infty$. However, the corresponding optimization problem is non-convex in α, as pointed out in Fujioka [2008].

Finally let us notice that the conservatism is Theorem 5 is more serious for very small h_0 since $\Phi_F(h_0)$ is close to the identity and γ is large. One might reduce the conservatism by modifying the modeling. Noting that

$$\Delta(\theta) = \int_0^\theta (e^{At} - I + I)\, dt = \theta I + \int_0^\theta (e^{At} - I)\, dt,$$

one can show that the exponential stability of T for $h \in [-\theta, \theta] + h_0$ if there exist a matrix $P > 0$ and α satisfying

$$\begin{bmatrix} \Omega_+(h_0, \theta) & I \\ \Psi(h_0) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega_+(h_0, \theta) & I \\ \Psi(h_0) & 0 \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & \alpha I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \Omega_-(h_0, \theta) & I \\ \Psi(h_0) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega_-(h_0, \theta) & I \\ \Psi(h_0) & 0 \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & \alpha I \end{bmatrix} < 0,$$

$$\frac{\sqrt{\alpha}}{2} \frac{\|A\|}{1 + \|A\|} \theta^2 e^{\|A\|\theta} \leq 1,$$

where

$$\Omega_\pm(h_0, \theta) := \Phi(h_0) \pm \theta \Psi(h_0).$$

In the derivation the following property Van Loan [1977] plays the key role:

$$\left\|e^{At} - I\right\| \leq t\, \|A\|\, e^{\|A\|t}, \tag{15}$$

although the detail is omitted. One drawback of this method is the dependence of conditions to be checked on θ which makes the algorithm more complicated.

Remark 4. Recently Hetel and Iung [2007] have proposed a method to construct a small polytope which outer approximates $\{\Phi_F(h) : h \in [h_\ell, h_u]\}$. It can help to reduce the conservatism as well.

6. CONCLUDING REMARKS

We have considered the stabilization problem via state feedback for sampled-data feedback control systems where the state is sampled aperiodically, motivated by widespread use of networked and embedded control systems.

We have proposed a stabilization algorithm by showing robustness of sampled-data systems against perturbation caused by variation of sampling intervals based on the small-gain framework. We have also discussed some directions for reducing the conservatism.

In this paper we have considered a simple sampled state feedback scenario, however, application to more practical synthesis problems are not hard.

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Proof of Theorem 5

We here prove that (10) holds for all $h \in [h_0, h_U]$. The proof for the interval $[h_L, h_0]$ is similar so it is omitted. Note that $H(h_0, \gamma) \subseteq [h_L, h_U]$.

Invoking Lemma 4 we have

$$\|\Delta(\theta)\| \leq \int_0^\theta \left\|e^{At}\right\| \, dt \leq \int_0^\theta e^{\mu(A)t} \, dt$$

when $\theta \geq 0$. If $\mu(A) = 0$

$$\|\Delta(\theta)\| \leq \theta.$$

Hence (10) holds as long as $\gamma\theta \leq 1$. This completes the proof for the case U1.

Let us next consider the case of $\mu(A) \neq 0$. In this case

$$\|\Delta(\theta)\| \leq \frac{e^{\mu(A)\theta} - 1}{\mu(A)}. \tag{.1}$$

Suppose that $\mu(A) < 0$. Noting that the right hand side goes to $-1/\mu(A)$ when θ tends to ∞. Hence (10) holds for all $\theta > 0$ if

$$-\frac{\gamma}{\mu(A)} \leq 1.$$

This completes the proof for the case U2.

Finally let us consider the case of $\mu(A) \neq 0$ and

$$-\frac{\gamma}{\mu(A)} > 1.$$

The small gain condition (10) holds for all $\theta > 0$ if

$$\gamma\frac{e^{\mu(A)\theta} - 1}{\mu(A)} \leq 1.$$

Noting that $1 + \gamma^{-1}\mu(A) > 0$ in this case, this condition turns to

Case A) If $\mu(A) > 0$

$$\mu(A)\theta \leq \log(1 + \gamma^{-1}\mu(A)).$$

Case B) If $\mu(A) < 0$

$$\mu(A)\theta \geq \log(1 + \gamma^{-1}\mu(A)).$$

Hence we have

$$\theta \geq \frac{1}{\mu(A)} \log(1 + \gamma^{-1}\mu(A)).$$

for both cases. This completes the proof for the case U3.