What is the 'Correct' Discretisation of the Linearised Navier-Stokes Equations for Feedback Flow Control?

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May 17, 2016

Why flow control?

- Road vehicles produce over 20% of world CO_2 emissions.

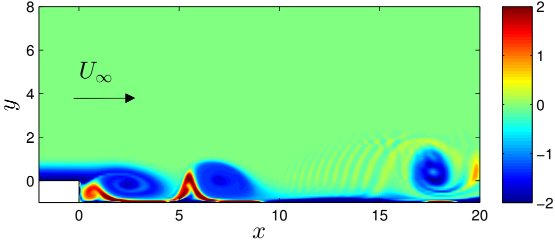

Figure: Backward facing step flow vorticity field.

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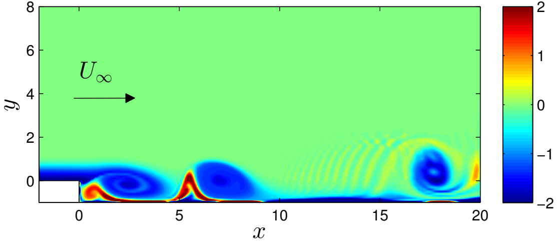
- Approximately two thirds of fuel consumption by goods transportation vehicles is consumed overcoming the aerodynamic drag arising from vortex shedding over the bluff rear end.



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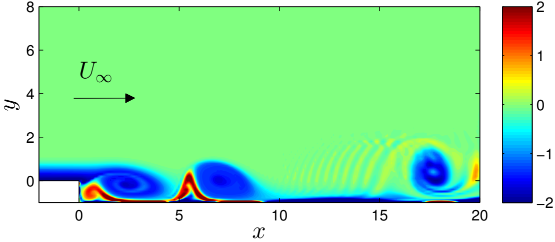

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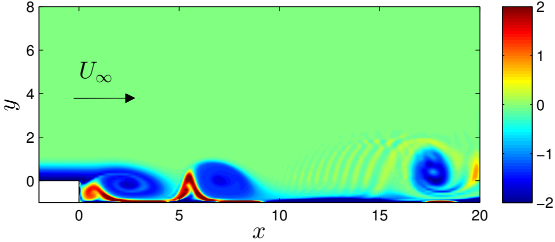
- How do we obtain plant models suitable for control design?



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How to discretise the governing equations?

- Equations governing fluid flow in typical road vehicle conditions are the incompressible Navier-Stokes equations; in order to deduce linear plant models we often start with the linearised versions of the equations:

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u}' = -\nabla p' + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}' \tag{1a}$$

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- For complex geometries, such as the backward facing step, finite difference approximations are the most straightforward way of discretising the governing equations.
- Does direct discretisation of (1) yield good models for control design?

Checkerboard instability

- Centered finite difference discretisations of (1) can yield non-physical sawtooth-like pressure field solutions - ‘checkerboard instability’.

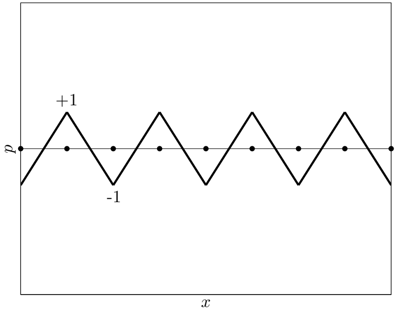
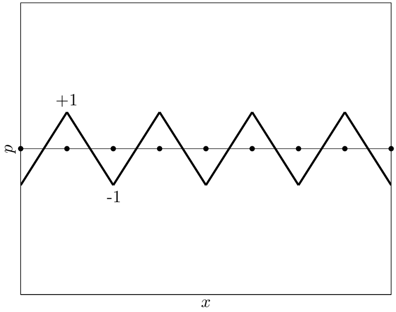


Figure: Non-physical, sawtooth-like pressure field satisfying discretised equations.

Checkerboard instability

- Centered finite difference discretisations of (1) can yield non-physical sawtooth-like pressure field solutions - ‘checkerboard instability’.
- Substituting divergence of momentum equation into continuity equation yields the pressure Poisson equation (PPE):

$$\nabla^2 p' = -\nabla \cdot \left(\mathbf{u}' \cdot \nabla \overline{\mathbf{u}}\right) - \nabla \cdot \left(\overline{\mathbf{u}} \cdot \nabla \mathbf{u}'\right) \quad (2)$$



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What pressure boundary conditions?

- The PPE formulation of the equations requires an additional pressure boundary condition.

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- But what condition? Some disagreement in the literature[1,2,3].

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- Do pressure boundary conditions alter the system dynamics?

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Pseudospectra for channel flow models

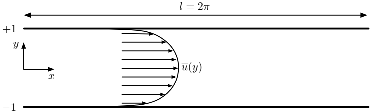

Figure: 2D channel flow geometry.

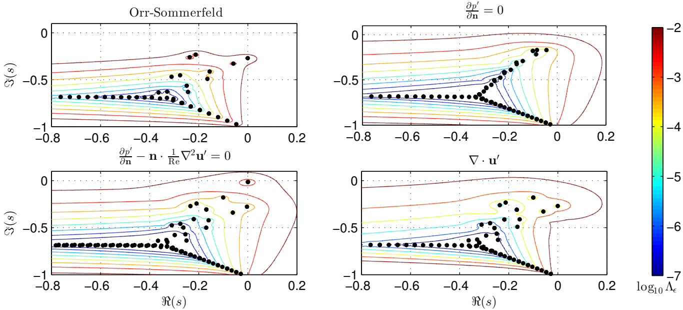

Figure: Model pseudospectra, $\Lambda_\epsilon := \{s \in \mathbb{C} : \underline{\sigma}\,(C(sI - A)B) \leq \epsilon\}$.

Frequency response

Wall transpiration input, $v(y = 1, t)$, wall pressure output, $p(y = -1, t)$

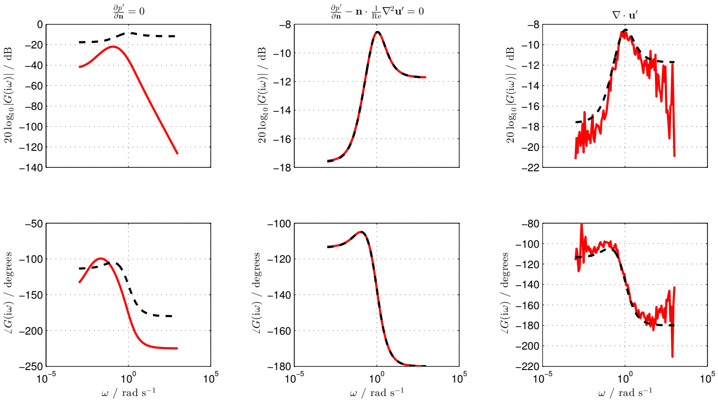

Figure: Frequency response of channel flow models.

Conclusions

- Whilst some PPE formulations appear to differ when considering only their spectra and pseudospectra, they may actually suffice as a plant model for controller design when appropriate inputs and outputs are chosen.

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- Whilst some PPE formulations appear to differ when considering only their spectra and pseudospectra, they may actually suffice as a plant model for controller design when appropriate inputs and outputs are chosen.
- The differences between formulations may, however, cause problems when they're being used for time-marching simulation purposes.
- **Models that are suitable for simulation purposes may not necessarily be suitable for control design, and vice versa.**