Observer-Based Residual Design for Nonlinear Systems

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Abstract: This paper presents a method for designing a full order observer for a class of nonlinear systems with unknown input in which the nonlinear functions satisfy Lipschitz conditions. The problem of detecting and isolating faults for this class of nonlinear systems are considered and the theoretical results are applied to a mass-spring-damper system in the presence of external disturbances and uncertainties to diagnose the sensor faults.

1. INTRODUCTION

The classical control methodologies are based on feedback in which the full state measurements should be available. In practical problems, measuring the states of a given system may be impossible and in many cases if it is possible the costs of such measurements are too expensive. Therefore, estimates of system states are required and are an important issue, particularly when a locally observable system is not linearly observable. For designing an observer for such nonlinear systems, it is endeavoured to convert them to an observable system or a canonical observer form. For nonlinear systems with unknown inputs where the measurement of all the input signals in a system is impossible, designing an observer enables one to estimate the states and consequently to design an appropriate control. In many cases, the uncertainties of certain parameters of a system may be modelled as unknown inputs to design an appropriate observer.

Many researchers have successfully designed various types of observers with unknown inputs for both linear and nonlinear systems. Based on these efforts, a reduced-order observer for linear systems with unknown inputs (Kudav et al. [1980]), a full order Luenberger type observer for linear systems (Darouach et al. [1994]), nonlinear systems with unknown inputs (Mondal et al. [2007]) and an unknown input observer for a certain class of nonlinear systems (Wannenberg [1990]) have been presented. All these works are based on the assumption that the nonlinearity depends only on the inputs and the outputs.

An unknown input observer for a more general class of nonlinear systems has been proposed by (Seliger and Frank [1998]) using a geometric approach. In addition, using various methods, many observers for nonlinear systems with unknown inputs have been presented in the last decade including an unknown input observer for nonlinear systems using a $H_\infty$ approach (Pertew et al. [2005]) and a nonlinear observer for a descriptive type of nonlinear systems with unknown inputs based on linear matrix inequality (LMI) approach by (Koeing [2006]) and (Chen and Saif [2006]). Koshkouei and Zinober [2004] have proposed a sliding mode observer for a class of nonlinear systems with uncertainties to stabilize the error estimation system with and without a Lipschitz nonlinearity.

Many alternative approaches have been considered to design unknown input observers for nonlinear systems using sliding mode control theory (Barbot et al. [2005]). Based on unknown input observer techniques, fault detection and isolation for many classes of nonlinear systems have been developed in recent years (Sharma et al. [2006]) and (Amato et al. [2002]). A reduced-order unknown input nonlinear observer based upon a transformation of coordinates to decouple a nonlinear system into two subsystems has also been developed (Koeing and Mammar [2001]).

This paper deals with designing an observer for a class of nonlinear systems in which the nonlinear functions satisfy the Lipschitz condition in terms of state and a method for diagnosing the faults is studied. The theoretical results are applied to a mass-spring-damper (M-S-D) model. In this paper $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the maximum and minimum eigenvalues of the symmetric matrix $(\cdot)$, respectively.

This paper is organised as follows: The mathematical description of the system is presented in Section 2. The design of nonlinear unknown input observer along with stability analysis of the error dynamics is addressed in Section 3. Section 4 provides the design of residual response for detecting the faults. The numerical example and conclusions are given in Sections 5 and 6, respectively.

2. SYSTEM DESCRIPTION

Consider the nonlinear system

\[
\begin{align*}
\dot{x} &= Ax(t) + Bu(t) + D\mu(t) + g(x, u, t) + K_a f_a(t) \\
y &= Cx(t) + K_s f_s(t)
\end{align*}
\]

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ represent the state, input and output vectors, respectively. Also $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $K_a \in \mathbb{R}^{n \times r}$ and $K_s \in \mathbb{R}^{p \times s}$
are known matrices, \( D \in \mathbb{R}^{n \times p} \) is the known input map disturbances and it is assumed to be a full rank matrix whilst \( \mu(t) \in \mathbb{R}^p \) is an unknown vector. The term \( D \mu(t) \) may actually describe additive disturbance and various modeling uncertainties such as noise, nonlinear or time-varying terms, linearisation and model reduction errors and parameter variations. In system (1), the term \( g(x, u, t) \) represents the known nonlinearity function, with \( f_s(t) \in \mathbb{R}^r \) and \( f_s(t) \in \mathbb{R}^s \) being failure on input and output sensors respectively.

Note that the output may include the control input \( B_y \) and the unknown input \( D_y \)
\[
y(t) = Cx(t) + B_y u(t) + D_y \mu(t) + K_s f_s(t)
\]
This case is not considered here because the term \( D_y \mu(t) \) can be nulled by using a transformation of the output signal \( g(t) \), and the input map \( B_y \mu(t) \) is eliminated since it does not affect the generality of the discussion on the observer design (Patton and Chen [1991]).

**Assumptions:** The following conditions should be satisfied for the existence of an unknown input observer:

- Nonlinearity \( g(t, u, x) \) is assumed to be globally Lipschitz in \( x \) with Lipschitz constant \( \kappa \), i.e.
  \[
  \|g(x, u, t) - g(\tilde{x}, u, t)\| \leq \kappa \|x - \tilde{x}\| \tag{2}
  \]
- The system should satisfy the following rank conditions:
  \[
  \text{rank}(CD) = \text{rank}(D) \tag{3}
  \]

3. NONLINEAR UNKNOWN INPUT OBSERVER (NUIO) DESIGN

3.1 Observer design

An observer is defined as an unknown input observer (UIO) for the system, if its state estimation error vector \( \hat{e}(t) \), approaches zero asymptotically, regardless of the presence of the unknown input term in the system.

3.2 UIO design

A full-order UIO for the system (1) is in the following form
\[
\hat{z}(t) = Nz(t) + Ly(t) + Gu(t) + H^*g(\hat{x}, u, t)
\]
where matrices \( N \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{n \times p}, G \in \mathbb{R}^{n \times m}, H^* \in \mathbb{R}^{m \times n} \), and \( E \in \mathbb{R}^{r \times p} \) are designed later to achieve the UIO.

Consider the state estimation error system
\[
\hat{e}(t) = x(t) - \hat{x}(t) = z(t) - E\hat{y}(t)
\]
where \( \hat{e}(t) = \hat{x}(t) - \hat{z}(t) + EC\hat{x}(t) + EK_sf_s(t) \)
Assume that
\[
H = I_n + EC
\]
Then the error system (5) is
\[
\dot{e}(t) = Ne(t) + (HA - NH - LC)x(t) + (HB - G)u(t) + HD\mu(t) + Hg(t, u, x) - H^*g(\hat{x}, u, t) + HK_s f_s(t) + (E - NE - L)K_s f_s(t)
\]
If the following conditions hold,
\[
HD = 0 \tag{8}
\]
\[
G = HB \tag{9}
\]
\[
HA - NH - LC = 0 \tag{10}
\]
\[
H = H^* \tag{11}
\]
then equation (7) may be written as
\[
\dot{e}(t) = Ne(t) + H(g(x, u, t) - g(\hat{x}, u, t)) + HK_s f_s(t) + (E - NE - L)K_s f_s(t) \tag{12}
\]
From equation (8)
\[
ECD = D
\]
and the solution for \( E \) is
\[
E = -D(CE)^{-1} \tag{13}
\]
where \( (CD)^+ \) is the pseudo-inverse of \( CD \). However, in many cases \( CD \) is a full rank matrix and therefore \( E = -D(CE)^{-1} \). On the other hand, \( H \) is found by substituting \( E \) into (6) and \( G \) is obtained by substituting \( H \) into (9).

The observer gain \( K \) is found using the pole placement or LQR methods which may be used such that the matrix \( HA - KC \) is stable.

\( N \) is selected such that the following relation is satisfied
\[
K = L + NE \tag{14}
\]
Using the conditions (6) and (10) \( N \) is obtained
\[
N = HA - KC \tag{15}
\]
Now the nonlinear UIO (4) and the error dynamics (12) may be rewritten as
\[
\dot{z} = (HA - KC)z(t) + Ly(t) + Gu(t) + Hg(\hat{x}, u, t) \tag{16}
\]
\[
\dot{e}(t) = (HA - KC)e(t) + H(g(x, u, t) - g(\hat{x}, u, t)) + HK_s f_s(t) + (E - K)K_s f_s(t) \tag{17}
\]
Then substituting \( N \) given by (15) into (14) the gain \( L \) is obtained
\[
L = K(I_p + CE) - HA E
\]
3.3 Stability analysis of the error system

Assume that there is no actuator fault on the system, i.e. \( f_a(t) = 0 \) and \( f_s \neq 0 \) with \( E = K \) where \( HA = A_0 \). Hence equation (17) becomes
\[
\dot{e}(t) = (A_0 - KC)e(t) + H(g(x, u, t) - g(\hat{x}, u, t)) \tag{18}
\]
Note that as long as \( E = K \), then \( L = E - NE \), hence the sensor fault \( f_s(t) \) does not affect the error system.

Consider the Lyapunov equation
\[
V = e^TP e
\]
where \( P \) is a symmetric positive-definite (s.p.d.) matrix.
If the following algebraic Riccati equation (ARE) is satisfied, then the error dynamics stability is guaranteed,
\[ P(A_0 - KC) + (A_0 - KC)^T P + \epsilon P^2 + \frac{1}{\epsilon} \kappa^2 I = 0 \] (19)
where \( \epsilon > 0 \).

The time-derivative of Lyapunov equation is
\[
\dot{V} = e^T P \dot{e} + e^T Pe < e^T [(A_0 - KC)^T P + P(A_0 - KC)]e + (g(x, u, t) - g(\hat{x}, u, t))^T Pe + Pe^T (g(x, u, t) - g(\hat{x}, u, t))
\] (20)

Since for any matrices \( X, Y \) and any positive number \( \epsilon > 0 \),
\[
\epsilon \left( \frac{1}{\epsilon} X - Y \right)^T \left( \frac{1}{\epsilon} X - Y \right) \geq 0
\]
then
\[
X^T Y + Y^T X \leq \frac{1}{\epsilon} X^T X + \epsilon Y^T Y
\]

Hence for any \( \epsilon > 0 \)
\[
e^T P (g(t, x, u) - g(t, \hat{x}, u)) + (g(t, x, u) - g(t, \hat{x}, u))^T Pe \leq \epsilon e^T Pe + \frac{1}{\epsilon} \| g(t, x, u) - g(t, \hat{x}, u) \|^2
\]
\[
\leq \epsilon e^T Pe + \frac{1}{\epsilon} \kappa^2 \| x - \hat{x} \|^2
\]
\[
= \epsilon \| e \|^2 + \frac{1}{\epsilon} \kappa^2 \| e \|^2
\] (21)

From (20) and (21)
\[
\dot{V} < e^T [(A_0 - KC)^T P + P(A_0 - KC)]e + \epsilon P^2 + \frac{1}{\epsilon} \kappa^2 I e
\] (22)

So the error system (18) is asymptotically stable if \( P \) is the solution of the matrix inequality
\[
(A_0 - KC)^T P + P(A_0 - KC) + \epsilon P^2 + \frac{1}{\epsilon} \kappa^2 I < 0
\] (23)

Since \( A_0 - KC \) is a stable matrix, the algebraic Riccati equation (ARE)
\[
(A_0 - KC)^T P_0 + P_0 (A_0 - KC) = -Q
\] (24)

has the unique s.p.d. matrix solution where \( Q \) is an arbitrary s.p.d. matrix.

Assume that the solution \( P = P_0 \) (24) is also a solution of the matrix inequality (23). Then from (22) and (24)
\[
\dot{V} < \left( -\lambda_{\min}(Q) + \epsilon \lambda_{\max}(P) + \frac{1}{\epsilon} \kappa^2 \right) \| e \|^2
\] (25)

Therefore, \( \dot{V} < 0 \) if
\[
\epsilon \lambda_{\max}(P) - \epsilon \lambda_{\min}(Q) + \kappa^2 < 0
\] (26)

The possible solutions for \( \epsilon \) are
\[
\frac{\lambda_{\min}(Q) - \sqrt{\Delta}}{2 \lambda_{\max}(P)} < \epsilon < \frac{\lambda_{\min}(Q) + \sqrt{\Delta}}{2 \lambda_{\max}(P)}
\] (27)
provided that \( \Delta = \lambda_{\min}^2(Q) - 4 \lambda_{\max}^2(P) \kappa^2 > 0 \) or

\[
\kappa < \frac{\lambda_{\min}(Q)}{2 \lambda_{\max}(P)}
\] (28)

However, the condition (26) is satisfied if and only if
\[
\lambda_{\min}(Q) > \frac{\epsilon^2 \lambda_{\max}(P) + \kappa^2}{\epsilon}
\] (29)

Note that all these stability condition are equivalent.

**Remark:** One may select \( Q = Q_1 + \frac{\kappa^2}{\epsilon} I > 0 \) for the ARE (24)
\[
(A_0 - KC)^T P + P(A_0 - KC) = -Q_1 - \frac{\kappa^2}{\epsilon} I
\] (30)

where \( Q_1 \) is a s.p.d. matrix. Then the error system is stable if
\[
\epsilon < \frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)}
\] (31)

By suitable selections of the design parameters \( \epsilon \) and matrix \( Q_1 \), the stability condition (31) may be satisfied which guarantees the asymptotic stability of the error dynamics.

### 4. FAULT DETECTION AND ISOLATION (FDI)

**SCHEMES BASED ON UIO**

The FDI techniques are based on the generation of appropriate residual signals which have to be sensitive to faults themselves but independent of disturbances. The method may be based on disturbance decoupling principles. In this approach, uncertain factors in the system modelling or identification are considered as an unknown input or the disturbance of the nonlinear system model. The disturbance vector is unknown but its is normally assumed that its distribution matrix is known. Based on disturbance distribution matrix obtained by modelling or identification procedure, the unknown input can be decoupled from the residual. The principle of a NUIO is to make the state (or output) estimation error decoupled from the unknown inputs or disturbances. Since the residual is a weighted output estimation error, it may be decoupled from each disturbance (Watanabe and Himmelblau [1982]), (Patton et al. [1989]) and (Frank [1990]).

Define the residuals as
\[
r = w(z + e, y, u)
\] (32)

**Assumptions:** For all possible unknown inputs \( \mu \) and faults \( f = 0 \) the point \( e(t) = 0 \) must be at least a local asymptotic stable equilibrium point of equation (17)
\[
w(z(x, u) + e, y, u) = 0
\]
and when \( f \neq 0 \),
\[
w(z(x, u) + e, y, u) \neq 0
\]

From (17), it can be seen that only the fault \( f \) and not the unknown input (disturbance) \( \mu \) excite the dynamics of the estimation error and therefore the residual. The residual is moreover sensitive to any fault affecting the system dynamics or the measurements.

The residual vector is given by
\[
r(t) = S_1z(t) + S_2y(t)
\] (33)

Substituting (4) and (1) into (33) yields
\[ r(t) = S_1 e(t) + (S_1 H + S_2 C)x(t) + S_1 E K_s f_s(t) + S_2 K_s f_s(t) \]  
\hspace{1cm} \text{(34)}

From (34) if the following conditions hold
\[ S_1 H + S_2 C = 0 \]  
\hspace{1cm} \text{(35)}

\[ S_1 E K_s + S_2 K_s \neq 0 \]  
\hspace{1cm} \text{(36)}

then the residual signal may be presented as follows,
\[ r(t) = -S_1 e(t) + (S_1 E + S_2) K_s f_s(t) \]  
\hspace{1cm} \text{(37)}

Matrices \( S_1 \in \mathbb{R}^{n \times n} \) and \( S_2 \in \mathbb{R}^{n \times p} \) can be found satisfying the above conditions.

5. NUMERICAL EXAMPLE

A M-S-D system with two masses, springs and dampers which are connected serially, is considered for numerical studies. The system equation is in the form (1), with the following matrices
\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -981 & -9.8 & 981 & 9.8 \\ 0 & 0 & 0 & 1 \\ 981 & 9.8 & -1962 & -19.6 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \]
\[ C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K_a = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \quad K_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
and
\[ D = \begin{pmatrix} 0.35 & 0 \\ 0.73 & 0.35 \\ -0.35 & 0.35 \\ 0 & 0.73 \end{pmatrix} \]

For a NUIO design the following steps are considered:

(a) First the conditions (2) and (3) must be satisfied where the nonlinearity is considered as \( g(x, u, t) = [0 \ \sin(x_2) \ 0 \ 0.8 \sin(x_4)]^T \), and the unknown input is \( \mu(t) = [\sin(2t) \ 0.5 \cos(t)]^T \), hence
\[ \|g(x, u, t) - g(\tilde{x}, u, t)\| \leq \kappa \|x - \tilde{x}\| \]
with \( \kappa = 0.79 \) and
\[ \text{rank}(CD) = \text{rank}(D) = 2. \]

(b) Next the conditions (6)-(11) must be satisfied.

From (13), \( E \) is found
\[ E = \begin{pmatrix} -1 & 0 \\ -2.08 & 0 \\ 0 & -1 \\ -2.08 & -2.08 \end{pmatrix} \]
Then substitute \( E \) into (6), \( H \) is obtained
\[ H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2.08 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2.08 & -2.08 & 0 \end{pmatrix} \]

Note that from (11), \( H = H^* \). For obtaining \( G \), substitute \( H \) and \( B \) into (9):
\[ G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 \end{pmatrix} \]

In this case the observer gain \( K \) has been considered such as \( K = E \). Then substitute observer gain \( K \) into (15), \( N \) is obtained
\[ N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -978 & -7 & 981 & 4.9 \\ 0 & 0 & 1 & 0 \\ 983.1 & 2.8 & -1959.1 & -11.9 \end{pmatrix} \]

(c) Now by substituting \( N \) into (14), \( L \) is obtained
\[ L = \begin{pmatrix} 0 & 0 \\ -985.4 & 991.2 \\ 0 & 0 \\ 962.1 & -1986.8 \end{pmatrix} \]

(d) Finally, substitute all matrices \( N, L, G, K \) and \( H \) in (16) to obtain the NUI observer for the M-S-D system.

5.1 Residual Design

For designing the residual response, \( S_1 \) and \( S_2 \) the following procedure is considered. For the system (1) consider the residual in the form of (34). The residual is robust if (35) has been satisfied, where \( S_1 \in \mathbb{R}^{1 \times n} \) and \( S_2 \in \mathbb{R}^{1 \times p} \) are the suitable design parameters.

Assume that for the M-S-D system \( S_1 \) and \( S_2 \) have the following forms
\[ S_1 = (s_{a_1} \ s_{a_2} \ s_{a_3} \ s_{a_4}) \]  
\hspace{1cm} \text{(38)}

and
\[ S_2 = (s_{b_1} \ s_{b_2}) \]  
\hspace{1cm} \text{(39)}

Hence (35) is satisfied if
\[ s_{a_2} = 0 \]
\[ s_{a_4} = 1 \]
\[ s_{b_1} = 2.08s_{a_4} \]
\[ s_{b_2} = -s_{a_4} \]
The residual is sensitive to faults if (36) is satisfied. Suppose that \( K_a = (1 \ -1)^T \) and \( f_s = 0.2 \cos(2t) \), then (36) is satisfied if,
\[ -s_{a_1} + s_{a_3} + s_{b_1} - s_{b_2} \neq 0 \]

A possible solution for \( S_1 \) and \( S_2 \) with respect to (35) and (36) is,
\[ S_1 = (-1 \ 0 \ -2 \ 1) \]  
\hspace{1cm} \text{(40)}

and
\[ S_2 = (2.08 \ -1) \]  
\hspace{1cm} \text{(41)}

Finally the scalar residual response for this system is generated in the form of (37). As long as the error dynamics (17) is stable, then the residual response is stable, robust and sensitive to faults.

5.2 Simulation Results

Consider the above M-S-D where the masses are 1Kg and the length of springs are 1m. The simulation has been carried out with and without the sensor fault which shows that the proposed method works successfully.

Figure 1 shows the behaviour of the real states and their estimates for the nonlinear M-S-D system where the actuator fault is considered to be zero and the system is...
Fig. 1. The behaviour of states and their estimates for the nonlinear M-S-D system. 

affected only by sensor faults. Figure 1 also shows that the states estimates eventually tend to the real states in a short period of time.

Figure 2 shows the residual and error signals for the M-S-D system in the absence of the faults. In this case the residual only depends on the error dynamics and tends to zero. The lower trace of Figure 2 shows that in the absence of the faults, the error states tends to zero eventually which guarantees the stability of the error system.

Figure 3 presents the residual response when the sensor fault affects the system between 15 and 30 seconds. This figure also shows that out of this time the residual tends to zero.

6. CONCLUSION

In this paper the full order Luenburger-like observer for nonlinear systems with unknown inputs is presented. The observer is applicable to a class of uncertain nonlinear systems in which the nonlinearity satisfies the Lipschitz condition. Necessary and sufficient conditions for the existence of the observer have been provided. The observer can be used diagnosing the system faults. Conditions under which it is possible to isolate single or multiple faults have been given. The residual signal has been generated to detect the sensor and actuator faults. The residual signal were independent of any uncertainties and is only affected by faults. Finally, the theoretical results have been applied to a nonlinear M-S-D system by showing how to use the FDI schemes and also the effectiveness of the FDI schemes.

REFERENCES


Fig. 3. The action of the residual signal when the sensor fault affects the system.


